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Note.—Part 2 of this Journal is the 1961-27 Newsletter of the Hydraulics ivision.

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DISCUSSION

Journal of the
HYDRAULICS DIVISION
Proceedings of the American Society of Civil Engineers

WATER SURFACE PROFILES IN IRREGULAR NATURAL STREAMS

By Praxitelis A. Argyropoulos[1], F. ASCE

SYNOPSIS

A relatively simple and practical method is proposed for the computation f water surface profiles in natural streams or reservoirs. It is a safe and pecially advantageous method when several backwater profiles must be de- ermined in the channel.

The velocity head corrections have been taken into account. The effect of end losses, bridge-pier losses, and losses owing to change in shape of the ross section can be included when necessary. The method is based on the ssumption that the velocity of flow is not uniformly distributed over the area f any cross section.

INTRODUCTION

In the past, several papers have been presented for the computation of water ırface profiles and of backwater curves in artificial canals or natural streams. everal direct or indirect step (analytical, graphical, or semigraphical) meth- ls have been published including those of Ruhmann, J. A. Bresse, Schaffernack, olkmitt, Bakhmeteff, Dupuit, Nagaho Mononobe, Ming Lee, Jansen, F. Ramponi, ano, Escoffier, Grimm, H. R. Leach, and N. Krivoshein. Attention is also lled to the proposed method by Arthur A. Ezra to replace and to eliminate ial and error computations and the method by Joe M. Lara and Kenneth B.

Note.—Discussion open until December 1, 1961. To extend the closing date one onth, a written request must be filed with the Executive Secretary, ASCE. This paper part of the copyrighted Journal of the Hydraulics Division, Proceedings of the Ameri- n Society of Civil Engineers, Vol. 87, No. HY 4, July, 1961.

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Schroedor that is a trial and error procedure involving step computations. A complete list of references and investigations is given in the Appendix II.

Notation.—The letter symbols adopted for use in this paper are defined where they first appear and are arranged alphabetically, for convenience of reference in Appendix I.

THEORY

In order to have steady uniform flow, a constant area and shape of section must be maintained, and the mean velocity must be the same at all sections. The water depth and the bed slope must be constant. However, if any of these fundamental conditions and characteristics are changed, the flow becomes nonuniform or varied. Usually, in this case the cross sectional area, the velocity and the hydraulic slope vary from section to section. The sections are irregular, the bed has an uneven slope, and the roughness of the bed or the banks may be continually changing. So, the balance between the friction loss and the slope is distrubed and the surface line is not parallel to the bottom. On the other hand, the position of the surface curve must be computed for the limiting cases of nonuniform flow. In natural watercourses such a computation is necessary. The classical example of varied flow is that of a backwater curve produced by a dam, weir, or other structures impeding the flow. These structures will cause the water to back up the stream, thereby increasing the depth. In such cases, the important question how much the water will be raised at a given distance upstream from the point of obstruction or any other given cross section arises. Formulas purporting to answer this question have been developed and given in many text and reference books on hydraulics, but most of them are based on an assumed regular channel.

According to Bernoulli's Theorem, in steady flow with friction and other losses present, the sum of the elevation head, pressure head, velocity head, and total losses is constant along any streamline. That is

$$Z + \frac{P}{w} + a\frac{V^2}{2\,g} + h_a = K = \text{a constant} \qquad (1)$$

In Eq. 1, h_a includes all head losses in the reach (n) - (n-1) considered. They are friction losses between sections (n) and (n-1) and other losses (eddy losses, owing to contraction or enlargement of the cross section of the channel, bend losses and bridge-pier losses). Thus, h_a = friction losses + other losses.

The friction losses are presented by the integral:

$$\int_{X_0}^{X_1} S\,dX$$

in which S is the rate of loss of head caused by friction and, X_0, X_1 the distance of the cross sections (n) and (n+1) from a given origin, measured upstream.

Usually, in actual practice, the friction losses owing to roughness and irregularities of the walls of the channel are computed by Chezy's formula $V = C\sqrt{R\,S}$, in which the determination of the coefficient C by Manning is $C = \frac{1.486}{n} R^{1/6}$. So the friction loss of head can be taken equal to

$$S = \frac{V^2}{C^2 R} = \left(\frac{n}{1.486} \frac{V}{R^{2/3}}\right)^2 \quad \text{. . . . . (2)}$$

in which n is the Manning's coefficient of roughness; V represents the mean velocity and; and R is the hydraulic radius.

The loss of head caused by the variations of the cross sectional area in the different sections (n) - (n-1) of the considered reach can be expressed by the integral

$$\int_{X_o}^{X_1} S_o \, dX = \int_{X_o}^{X_1} r\, V^2 \, dX \quad \text{. . . . . (3)}$$

in which the coefficient r is a complex function of the area and its derivation dA/dX, that is from elements that are depended from the abscissas X.

To simplify the computation of the total losses caused by enlargement or contraction of natural streams, bend losses, and bridge-pier losses, it is possible to consider them as a percentage of the change in velocity heads between (n) and (n+1).

The empirical coefficient of Bazin or the coefficient of velocity,

$$a = \frac{\int V^2 \, dA}{V^2 A} \quad \text{. . . . . (4)}$$

is dependent on the rate of uniformity of the distribution of the velocity flow, the viscosity of the fluid and the form and the nature of the bed and the banks of the channel. Its value is usually slightly greater than unity. In case of a rectilinear canal with smooth walls, a = 1.01 - 1.10. In the case of rough walls a = 1.12. For natural streams, the value of the coefficient could possibly be a = 1.20 - 1.40. But in the practice, owing to the confined value of the velocity-head $V^2/(2\,g)$, and the small influence on it of a, it is generally accepted that a = 1.00.

The total friction head for the reach under consideration will be

$$h_a = \int_{X_o}^{X_1} S \, dX + \int_{X_o}^{X_1} r\, V^2 \, dX = \int_{X_o}^{X_1} (S + r\, V^2) dX \quad \text{. . . . (5)}$$

and substituting in Eq. 1 there results

$$Z + \frac{P}{w} + a \frac{V^2}{2\,g} + \int_{X_o}^{X_1} (S + r\, V^2) dX = K \quad \text{. . . . . (6)}$$

Differentiating Eq. 6

$$dZ + \frac{dP}{w} + a \frac{V\, dV}{g} + (S + r\, V^2)\, dX = 0 \quad \text{. . . . . (7)}$$

or

$$dZ = - \left[\frac{dP}{w} = a \frac{V\, dV}{g} + (S + r\, V^2) dX\right] = 0. \quad \text{. . . . . (8)}$$

But here depths and not heights are represented, therefore, Eq. 8 can be written

$$dZ = \frac{dP}{w} + a\frac{V\,dV}{g} + \left(S + r\,V^2\right)dX \quad \ldots\ldots\ldots\ldots (9)$$

For a free water surface, the pressure P is equal to the atmospheric pressure, P_a. That is $P = P_a = a =$ a constant quantity and $dP = 0$. So Eq. 3 becomes

$$\frac{dZ}{dX} = \frac{d}{dX}\left(a\,\frac{V^2}{2\,g}\right) + \left(S + r\,V^2\right) \quad \ldots\ldots\ldots\ldots (10)$$

and

$$Z_{n-1} - Z_n = \left(-\,a_{n+1}\,\frac{V_{n+1}^2}{2\,g} + a_n\,\frac{V_n^2}{2\,g}\right) + \int_{X_o}^{X_1}\left(S + r\,V^2\right)dX \quad \ldots (11a)$$

If $a_{n+1} = a_n = a$,

$$Z_{n+1} - Z_n = \frac{a}{2\,g}\left(-\,V_{n+1}^2 + V_n^2\right) + \int_{X_o}^{X_1}\left(S + r\,V^2\right)dX \quad \ldots (11b)$$

Because the discharge $Q = V\,A$ and the velocity $V = Q/A$, Eq. 11b can be written

$$Z_{n+1} - Z_n = a\frac{Q^2}{2\,g}\left[-\,\frac{1}{A_{n+1}^2} + \frac{1}{A_n^2}\right] + Q^2\left[\int_{X_o}^{X_1}\left(\frac{P}{C^2\,A^3} + \frac{r}{A^2}\right)dX\right] \quad \ldots (12)$$

because

$$S = \frac{V^2}{C^2\,R} = \frac{V^2\,P}{C^2\,A} = \frac{Q^2\,P}{C^2\,A^3} \quad \ldots\ldots\ldots\ldots (13)$$

According to Eq. 12, if for each discharge Q the elevation Z_n at the end o the chosen reach is known, the elevation at the section (n+1) is a function o Q, A, and P because the general topography of the channel and the roughnes of the walls are not changed with time. For the same discharge, at Z_n corre sponding values of the area A, wetted perimeter P and coefficients C and r ar also fixed. Under the same presuppositions, it is possible to say that in steady, nonuniform flow the discharge Q in a given section is a function of th free water surface elevation.

In natural watercourses the area A, wetted perimeter P, and the coefficie of resistance C in Chezy's formula, that varies with R, the roughness and prob ably the shape of the cross section usually are constantly changing from sec tion to section in the considered reach. However, it is very difficult to estab lish the relation existing among them.

The evaluation of the integral

$$\int_{X_o}^{X_1} h_a + \int_{X_o}^{X_1} \left(S + r\,V^2\right) dX = Q^2 \left[\int_{X_o}^{X_1} \frac{P}{C^2\,A^3}\, dX + \int_{X_o}^{X_1} \frac{r}{A^2}\, dX \right] \quad (14)$$

would be possible if the change of the elements A, P, and the coefficient r is taking place in the length $\Delta X = X_1 - X_o$ with a well established manner. That is the functions: $A - f_1(X)$, $P = f_2(X)$, and $r = f_3(X)$ are known. But, in irregular natural streams, the cross sections are not changing according to any mathematical law. Consequently, the preceding functions are not possible to be established. Only by approximation is it possible to realize the evaluation of the integrals $\int S\, dX$ and $\int r\, V^2\, dX$, that is, by dividing the considered reach into a number m of equal lengths. For each cross section the area A and wetted perimeter P are computed. However, if $y_o, y_1, y_2 \ldots y_m$ is the value of expression $(1/C^2)\ (P/A^3) = y$ for each section, then:

$$\int_{X_o}^{X_1} \frac{1}{C^2} \frac{P}{A^3}\, dX = \frac{X_1 - X_o}{m} \left[2\left(y_1 + y_3 + y_5 + \ldots + y_{m-1}\right) + \frac{y_o + y_m}{6} - \frac{y_1 + y_{m+1}}{6} \right] \quad (15)$$

in which $X_1 - X_o = \Delta X$ is the distance between the sections (n) and (n+1) considered.

The approximation obtained is better as the number of the equal lengths m is increased. In this case, because $C = \frac{1.486}{n} R^{1/6}$, the definition of Manning's coefficient of roughness, n, is necessary. Analogous is the procedure for the integral

$$\int_{X_o}^{X_1} \frac{r}{A^2}\, dX$$

However, the solution of the problem for computing water surface profiles in the case of an irregular channel becomes easier by using an alternate value, $S_m\, \Delta X$, in which S_m is the arithmetic mean of the values S_n and S_{n-1}. In this case, if the curve of S versus X is concave upward

$$S_m\, \Delta X < \int_{X_o}^{X_1} S\, dX < \frac{S_n + S_{n+1}}{2}\, \Delta X. \quad (16)$$

and

$$S_m\, \Delta X > \int_{X_o}^{X_1} S\, dX > \frac{S_n + S_{n+1}}{2}\, \Delta X. \quad (17)$$

if the preceding curve is concave downward.

The inaccuracy is small by using $S_m \Delta X$ instead of

$$\int_{X_o}^{X_1} S\, dX$$

when comparing other inaccuracies involved such as the choice of values of the different hydraulic elements, as for instance that of Manning's coefficient of roughness. Also, the inaccuracy is small by using

$$S_m = \frac{\left(S_n + S_{n+1}\right)}{2} \qquad (18)$$

instead of

$$S'_m = \frac{V_m^2}{C_m^2\, R_m} = \frac{Q^2}{C_m^2\, R_m\, A_m^2} = \frac{n_m^2\, V_m^2}{2.2\, R_n^{4/3}} \qquad (19)$$

in which

$$C_m = \frac{C_n + C_{n+1}}{2} \qquad (20)$$

$$R_m = \frac{R_n + R_{n+1}}{2} \qquad (21)$$

and

$$A_m = \frac{A_n + A_{n+1}}{2} \qquad (22)$$

The difference between S_m and S'_m is small, and there being some slight inaccuracy in the expression $\Delta = \Delta X\left(S_m - S'_m\right)$ the result is within normal limits of permissible errors. Also, it must be noted that

$$S_m = \frac{n^2}{2.2} \frac{V_m^2}{R_m^{4/3}} = \frac{\left(\frac{V_n + V_{n+1}}{2}\right)^2}{\left(\frac{R_n + R_{n+1}}{2}\right)^{4/3}} \frac{n^2}{2.2} = 0.63\, \frac{n^2}{2.2} \frac{\left(V_n + V_{n+1}\right)^2}{\left(R_n + R_{n+1}\right)^{4/3}} \qquad (23)$$

That is

$$S_m = \left[0.534\, n\, \frac{\left(V_n + V_{n+1}\right)}{\left(R_n + R_{n+1}\right)^{2/3}}\right]^2 \qquad (24)$$

On the other hand, for the eddy losses (as previously mentioned) it is a general practice to determine them by assigning a certain percentage, a_o, of the change in velocity head between the sections (n) and (n-1). That is

$$a_o\left(\frac{V_n^2}{2\,g} - \frac{V_{n+1}^2}{2\,g}\right)$$

For accelerated flow, $V_n > V_{n+1}$

For retarded flow, $V_n < V_{n+1}$

in which $a_o = 0.5$ in case of an abrupt expansion or contraction; $a_o = 0.2$ for a gradually diverging reach and; and $a_o = 0$ to 0.1 for a gradually converging reach. Therefore, it is possible to write

$$\Delta Z = Z_{n+1} - Z_n = \frac{a}{2\,g}\left(-V_{n+1}^2 - V_n^2\right) + \frac{S_n + S_{n+1}}{2}\,\Delta X + \frac{a_o}{2\,g}\left(-V_{n+1}^2 + V_n^2\right)$$

$$= \frac{1}{2\,g}\,(a + a_o)\left(-V_{n+1}^2 + V_n^2\right) + \frac{\Delta X}{2}\left(\frac{V_n^2}{C_n^2\,R_n} + \frac{V_{n+1}^2}{C_{n+1}^2\,R_{n+1}}\right) \quad \ldots\ldots (25)$$

or

$$Z = \left(\frac{a + a_o}{2\,g}\right) Q^2\left(\frac{1}{A_{n+1}^2} + \frac{1}{A_n^2}\right) + \frac{\Delta X}{2}\,Q^2\left(\frac{1}{C_n^2\,A_n^2\,R_n} + \frac{1}{C_{n+1}^2\,A_{n+1}^2\,R_{n+1}}\right) \quad \ldots (26)$$

Considering that the ruggosity factor is constant in the reach (n) - (n-1), as $C = \frac{1.486}{n} R^{1/6}$, it is possible to have

$$\Delta Z = \left(\frac{a + a_o}{2\,g}\right) Q^2\left(\frac{1}{A_{n+1}^2} + \frac{1}{A_n^2}\right) + \frac{\Delta X\,Q^2\,n^2}{4.4}\left(\frac{1}{A_n^2\,R_n^{4/3}} + \frac{1}{A_{n+1}^2\,R_{n+1}^{4/3}}\right) \quad \ldots (27)$$

The discharge Q is constant and the elements n, ΔX, and the acceleration due to gravity g are known. In Eq. 27 the expression of the form: $\Delta X\,Q^2\,n^2/4.4$ may be presented by B_1 and the expression $\left(\frac{a + a_o}{2\,g}\right) Q^2$ by B_2. Eq. 27 can be rewritten as

$$\Delta Z = B_2\left(\frac{1}{A_{n+1}^2} + \frac{1}{A_n^2}\right) + B_1\left(\frac{1}{A_n^2\,R_n^{4/3}} + \frac{1}{A_{n+1}^2\,R_{n+1}^{4/3}}\right) \quad \ldots\ldots (28a)$$

or

$$\Delta Z = \frac{1}{A_n^2}\left(\frac{B_1}{R_n^{4/3}} + B_2\right) + \frac{1}{A_{n+1}^2}\left(\frac{B_1}{R_{n+1}^{4/3}} - B_2\right) \quad \ldots\ldots\ldots (28b)$$

For the sections (n) and (n+1) considered, the area A, the wetted perimeter P, the hydraulic radius R, and consequently, the expressions $\left(\frac{a + a_o}{2\,g}\right)\frac{Q^2}{A^2}$ and $\frac{\Delta X\,Q^2\,n^2}{4.4\,A^2\,R^{4/3}}$ are functions of the depth of flow. Also, they are functions of the elevation of the water surface above a chosen datum. So, the expressions $\frac{1}{A_n^2}\left(\frac{B_1}{R_n^{4/3}} + B_2\right)$ and $\frac{1}{A_{n+1}^2}\left(\frac{B_1}{R_{n+1}^{4/3}} - B_2\right)$ are functions of the hydraulic depth.

If the expression $\frac{1}{A_n^2}\left(\frac{B_1}{R_n^{4/3}} + B_2\right)$ is denoted by $F(Z_n)$ and the expression $\frac{1}{A_{n+1}^2}\left(\frac{B_1}{R_{n+1}^{4/3}} - B_2\right)$ by $F(Z_{n+1})$, Eq. 28b can be written:

$$\Delta Z = Z_{n+1} - Z_n = F(Z_n) + F(Z_{n+1}) \quad \ldots\ldots\ldots\ldots (29)$$

It would be possible to have a direct step solution of Eq. 29 by a graphical or semigraphical means but it is the writer's opinion that this procedure for computing and tracing the water surface profiles is more laborious and difficult than the trial and error method. For this reason, a simple and practical step by step method is applied in tracing the water surface profiles.

It must be noted that to obtain the necessary data for the computations a survey of the irregular channel should be made and the shape of the different cross sections with elevations determined to a sufficient height for use in defining the channel geometry. The cross sections should be plotted, the area at each section determined by a planimeter, and the perimeter measured for the different water levels. This survey should also include the measurement of the considered reaches and the collection of the necessary data for the selection of the proper value of the coefficient of roughness, n. The results of the computations are then tabulated.

Because the discharge, Q, length ΔX, area A_n, hydraulic radius R_n, and the coefficients a, a_o and n are known, the values of the B_1 and B_2 can be found. Also, the value of the function $F(Z_n)$ in Eq. 29 can be found, thus,

$$F(Z_n) = - F(Z_{n-1}) + \Delta Z = \frac{1}{A_n^2}\left(\frac{B_1}{R_n^{4/3}} + B_2\right) \quad \ldots\ldots\ldots (30)$$

A value is assumed for ΔZ. By performing the required computations, it will be possible to check up if this value is correct.

Simultaneously, $Z_{n+1} = Z_n + \Delta Z$; the area A_{n+1}, hydraulic radius R_{n+1}, and the value of the expression $\frac{1}{A_{n+1}^2}\left(\frac{B_1}{R_{n+1}^{4/3}} - B_2\right)$ can be found. The sum, $F(Z_{n+1}) + F(Z_n)$ is checked to determine whether the assumed ΔZ is correct. Otherwise, the computations are repeated. Usually, two or three trial computations are required to obtain the correct fall.

APPENDIX I.—NOMENCLATURE

The following symbols are adopted for use in this paper.

A = area of the cross section; in square feet

a = coefficient of Bazin or velocity coefficient;

a_o = ratio of eddy losses to velocity-head change in any two consecutive sections;

ΔX = length of a channel reach, in feet;

g = acceleration due to gravity, in feet per second per second;

h_a = total head loss of flow in a section above a given datum in feet;

K = a constant;

m = number of equal lengths. When used as a subscript it denotes the mean value;

n = Manning's coefficient of roughness;

P = wetted perimeter of a cross section, in feet;

Q = discharge, in cubic feet per second;

R = hydraulic radius, in feet;

r = a coefficient;

S = rate of loss of head by friction at any section;

V = mean velocity of flow in a cross section, in feet per second;

X = abscissas indicating the distance along the channel bottom, in feet;

w = specific weight of water; and

Z = water surface elevation at any section, in feet.

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END DEPTH AT A DROP IN TRAPEZOIDAL CHANNELS

By M. H. Diskin[1]

SYNOPSIS

The momentum equation is used to derive a general equation for the end depth in prismatic channels in mild slopes at an abrupt drop. The equation is solved directly for exponential channels and in a tabular form for trapezoidal channels. Experiments on two trapezoidal channels are reported and the results are compared with the theoretical values of end depth derived by use of the momentum equation.

INTRODUCTION

The design of the upstream portion of a vertical drop in open channels on mild slopes is usually done according to one of two methods. One is to build, near the end section, a low weir or contracting walls, the purpose of which is to maintain the depth of water at or near the normal depth in the channel, thus avoiding the drawdown water surface curve and the high velocities accompanying it. The second method of design is to leave the channel section unaltered up to the drop and to protect a certain length of the channel before the drop by concrete or stone lining to resist the erosion due to the high velocities developed.

One of the factors which enter the second method of design and in particular the design of the stilling basin below the fall, is the end depth, that is, the depth of water at the end section of the upstream channel just before the drop. Know-

Note.—Discussion open until December 1, 1961. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Hydraulics Division, Proceedings of the American Society of Civil Engineers, Vol. 87, No. HY 4, July, 1961.

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ledge of the end depth is also useful in cases for which the end depth in existing drops is used for flow estimation, the California Pipe Method of flow measurement[2,3] being an example for such use. The end depth has been fairly extensively investigated for the case of rectangular channels[4,5,6,7,8,9,10,11,12,13,14] and to a smaller extent for circular channels,[2,3,15,16] but little or no work has been done on the end depth in trapezoidal channels.

Use is made herein of the momentum equation to develop a relationship between the end depth and the apparent critical depth and hence between the end depth and the discharge. The equations obtained are solved numerically for particular cases of trapezoidal channels of various side slopes and the results obtained are presented in tables and graphs. Finally, some experimental results are presented and compared with the theoretical results obtained previously.

Notation.—The letter symbols adopted for use in this paper are defined where they first appear, in the illustrations or in the text, and are arranged alphabetically, for convenience of reference, in the Appendix.

CONDITIONS NEAR THE END SECTION

The water surface curve near a drop in a channel on a mild slope is a drawdown curve of the M 2 type (Fig. 1). It starts at the normal depth upstream and ends, at the end section, with a critical depth, that is, a depth giving a minimum specific energy for the given discharge.

The fact that the end section is a section of minimum specific energy is a result of the fact that in the drawdown curve, the mean velocity of flow in-

2 "The California-Pipe Method of Water Measurement," by B. R. VanLeer, Engineering News-Record, August 3, 1922.

3 "The California-Pipe Method of Water Measurement," by B. R. VanLeer, Engineering News-Record, August 21, 1924.

4 "Open End Flume Water Meter Based on Exponential Equation," by J. W. Ledoux, Engineering News-Record, September 25, 1924, p. 505.

5 "Analyzing Hydraulic Models for the Effects of Distortion," by M. P. O'Brian, Engineering News-Record, September 15, 1932.

6 "Discharge Characteristics of the Free Overfall," by H. Rouse, Civil Engineering, April, 1936, p. 257.

7 "Discharge Over a Free Fall," by W. B. Langbein, Civil Engineering, May, 1937, p. 349.

8 Discussion by H. Rouse of "Discharge Over a Free Fall," by W. B. Langbein, Civil Engineering, July, 1937, p. 518.

9 "Energy Losses at the Base of a Free Overfall," by W. L. Moore, Transactions, ASCE, Vol. 108, 1943, p. 1343.

10 "Hydraulic Drop as a Function of Velocity Distribution," by C. G. Edson, Civil Engineering, December, 1954, p. 814.

11 "Hydraulics of the Free Overfall," by A. Fathy and M. Shaarawi, Proceedings, ASCE, Vol. 80, Proceedings-Separate No. 564, December, 1954.

12 "Rapidly Varied Flow Offers Wide Scope for Investigation," by J. C. I. Doodge (Containing bibliography on free overfall), Civil Engineering, February, 1955, p. 100

13 "Influence of Slope and Roughness on the Free Overfall," by J. W. Delleur, J. C. I. Doodge, and K. W. Gent, Proceedings, ASCE, Vol. 82, No. HY 4, August, 1956.

14 "Characteristics of Flow over Terminal Weirs and Sills," by P. K. Kandaswamy and H. Rouse, Proceedings, ASCE, Vol. 83, No. HY 4, August, 1957.

15 "Discharge of Pipes Flowing Partly Full," by C. Rohwer, Civil Engineering, Vol 13, 1943, p. 488.

16 "Flow Measurements in Circular Channels," by M. H. Diskin, M.Sc. Thesis (in Hebrew), Technion, Haifa, Israel, September, 1958.

creases continuously in the downstream direction. The increase in the mean velocity results in a continuous increase in the slope of the energy line, a process which continues up to the end section. The specific energy is defined as the vertical distance between the energy line and the invert of the channel, as the slope of the channel is constant, the specific energy diminishes in the downstream direction and the end section is thus a section of minimum specific energy. The end depth, which is the true critical depth, is smaller than the apparent critical depth (h_c) defined by the equation

$$\frac{Q^2 B_c}{g A_c^3} = 1 \qquad (1)$$

in which Q is the discharge, B denotes the width of channel at water surface, A refers to the cross-sectional area of flow, and the subscript c denotes that the quantity is related to apparent critical section. Fig. 1 is based on the assumptions of uniform velocity distribution and straight parallel streamlines,

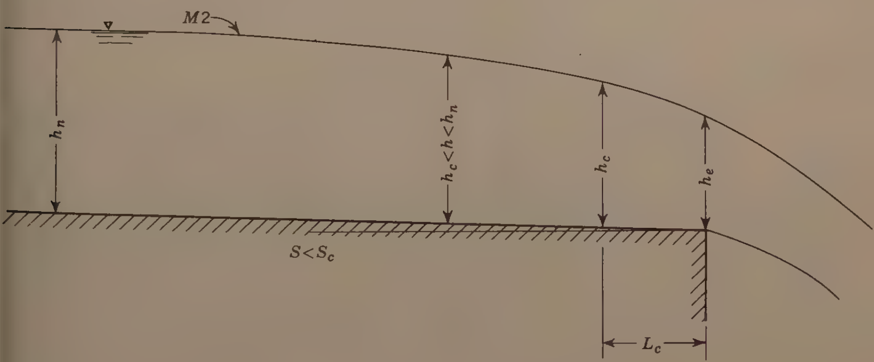

FIG. 1

giving hydrostatic pressure distribution. The convergence of the stream lines at the drawdown curve near the drop is conducive to the production of uniform velocity distribution, but on the other hand, the curvature of the lines produces a non-hydrostatic pressure distribution. Near the end section the curvature is much increased so as to give, finally, at the end section itself, a pressure distribution which is zero on the entire periphery of the nappe. This is in agreement with the physical fact of the nappe being in contact with the atmosphere on its periphery, as it springs clear off the upper channel. Because of the non-hydrostatic pressure distribution the depth can be smaller than that given by Eq. 1.

As the end depth is smaller than the apparent critical depth and the water surface curve near the end depth is dropping and strongly curved, there is a section, a short distance (L_c) upstream from the end section, at which the

depth is equal to that given by Eq. 1. This section will be called the apparent critical section.

THE MOMENTUM EQUATION

Conditions near the end section are those of a local phenomenon in which it is possible to use the momentum equation. The momentum equation will be written for a control volume bounded by the end section and by the apparent critical section, and it will be used to derive a relationship between the depths in the two sections. The following assumptions are made in applying the momentum equation:

1. The pressure distribution at the apparent critical section is hydrostatic. The basis for this assumption is the observed fact that this section is located just before the strongly curved portion of the water surface curve. Also, measurements of pressure distributions in rectangular channels[5,8,11] support this assumption.

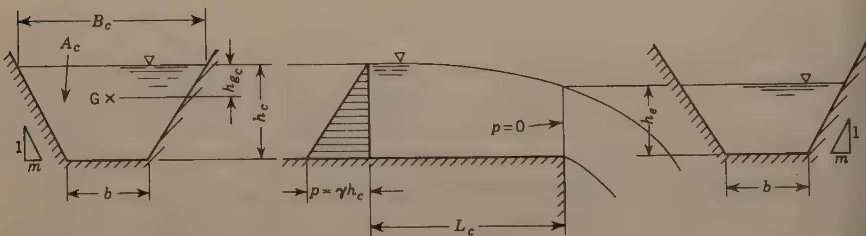

FIG. 2

2. The pressure distribution at the end section is zero. As the pressure at all points on the periphery of the end section is zero, the pressure at points in the section can not be far from zero. Reported measurements of pressure distributions at the end section of rectangular channels[8,11] indicate that the residual pressures are small. Hunter Rouse,[8] F. ASCE, reports positive residual pressures, while A. Fathy and M. Shaarawi[11] report positive pressures over part of the section and negative over another part. It appears that the sum of the pressure forces, which is the quantity required for the momentum equation, is indeed small.

3. The velocity distributions at both the end section and the apparent critical section are uniform. The two sections are located in zones of converging flow lines which usually produce uniform velocity distribution so that the assumption is reasonable. Allowance could be made for non-uniform distributions by introducing the momentum transport coefficient (β) into the momentum equation. However, as there is no information about values of these coefficients, it is felt that no advantage could be gained by introducing the coefficient and then assuming values of $\beta_c = \beta_e = 1.0$ at a later stage in the development.

4. The distance (L_c) between the end section and the apparent critical section is short so that the shear forces on the body of water between the two sections could be neglected.

5. The slope of the channel is small so that the component of the weight of water between the sections is negligible.

It should be noted that the two forces, assumed to be negligible in assumptions 4 and 5, act to opposite directions so that their combined effect must be very nearly zero even if the forces are not entirely negligible.

The momentum equation, for the body of water between the end section and the apparent critical section (Fig. 2), can, on the basis of the preceding assumptions, be written in the form

$$\gamma h_{g_c} A_c - 0 = \left(\frac{\gamma}{g}\right) Q \left(V_e - V_c\right) \quad \text{. . . . . (2)}$$

in which γ denotes the specific weight of the fluid, h_g refers to the depth of the centroid of section below the water surface, V is the mean velocity in the section, and the subscript e denotes that the quantity is related to the end section. Rearranging Eq. 2 and substituting $V = Q/A$, yield

$$h_{g_c} A_c = \frac{Q^2}{g}\left(\frac{1}{A_e} - \frac{1}{A_c}\right) \quad \text{. . . . . (3)}$$

At the critical section

$$\frac{Q^2}{g} = \frac{A_c{}^3}{B_c} \quad \text{. . . . . (4)}$$

Substituting this relationship in Eq. 3, rearranging and solving for the ratio (A_e/A_c)

$$\frac{A_e}{A_c} = \frac{1}{1 + h_{g_c}\dfrac{B_c}{A_c}} \quad \text{. . . . . (5a)}$$

or

$$\frac{A_e}{A_c} = \frac{1}{1 + \dfrac{h_{g_c}}{\bar{h}_c}} = \frac{\bar{h}_c}{\bar{h}_c + h_{g_c}} \quad \text{. . . . . (5b)}$$

in which $\bar{h}_c$ is the mean depth at the apparent critical section.

Eq. 5b defines the relationship between the area of the end section (A_e) and the area of the apparent critical section (A_c). If the geometry of the section is known, this also defines the relationship between the end depth (h_e) and the apparent critical depth (h_c). Because the relationship between the apparent critical depth and the discharge is known (Eq. 1), it is possible also to obtain a relationship between the end depth and the discharge.

SOLUTION FOR EXPONENTIAL CHANNELS

An exponential channel is defined here as a channel for which the relationship between the cross-sectional area (A) and the depth (h) can be expressed by the exponential equation

$$A = K\,h^n \quad \text{(6)}$$

in which K is the coefficient in the equation for the area in exponential channels. It can be shown[17] that for exponential channels the mean depth is given by

$$\bar{h} = \frac{1}{n}\,h \quad \text{(7)}$$

and the depth to the centroid of the section (h_g) by

$$h_g = \frac{1}{n+1}\,h \quad \text{(8)}$$

Substituting Eqs. 7 and 8 into Eq. 5, the ratio of area of end section to area of apparent critical section in exponential channels may be expressed as

$$\frac{A_e}{A_c} = \frac{n+1}{2\,n+1} \quad \text{(9)}$$

Using Eq. 6 the expression obtained for the ratio of end depth to apparent critical depth in exponential channels is

$$\frac{h_e}{h_c} = \left(\frac{n+1}{2\,n+1}\right)^{1/n} \quad \text{(10)}$$

The relationship between values of (A_e/A_c) and (h_e/h_c) and values of the exponent n, computed by Eqs. 9 and 10, is given in Table 1 and presented graphically in Fig. 3.

In particular, the values of the depths ratio for rectangular, parabolic, and triangular channels are

$$\frac{h_e}{h_c} = \frac{2}{3}\;1.0 = 0.667 \quad \text{(11)}$$

for a rectangular channel (n = 1.0)

$$\frac{h_e}{h_c} = \left(\frac{2.5}{4.0}\right)^{1/1.5} = 0.731 \quad \text{(12)}$$

for a parabolic channel (n = 1.5), and

$$\frac{h_e}{h_c} = \left(\frac{3}{5}\right)^{1/2} = 0.775 \quad \text{(13)}$$

for a triangular channel (n = 2.0).

SOLUTION FOR TRAPEZOIDAL CHANNELS

Considering the geometry of a trapezoidal channel of bottom width b and side slopes m (Fig. 4), expressions may be developed for the mean depth $(\bar{h})$ a

17 "Hydraulic Jump in Trapezoidal Channels," by M. H. Diskin, Water Power, January, 1961, p. 12.

TABLE 1

n	$\frac{A_e}{A_c}$	$\frac{h_e}{h_c}$
0.9	0.6786	0.6500
1.0	0.6667	0.6667
1.1	0.6562	0.6818
1.2	0.6471	0.6958
1.3	0.6389	0.7085
1.4	0.6316	0.7202
1.5	0.6250	0.7310
1.6	0.6190	0.7410
1.7	0.6136	0.7503
1.8	0.6087	0.7590
1.9	0.6042	0.7671
2.0	0.6000	0.7746
2.1	0.5962	0.7817

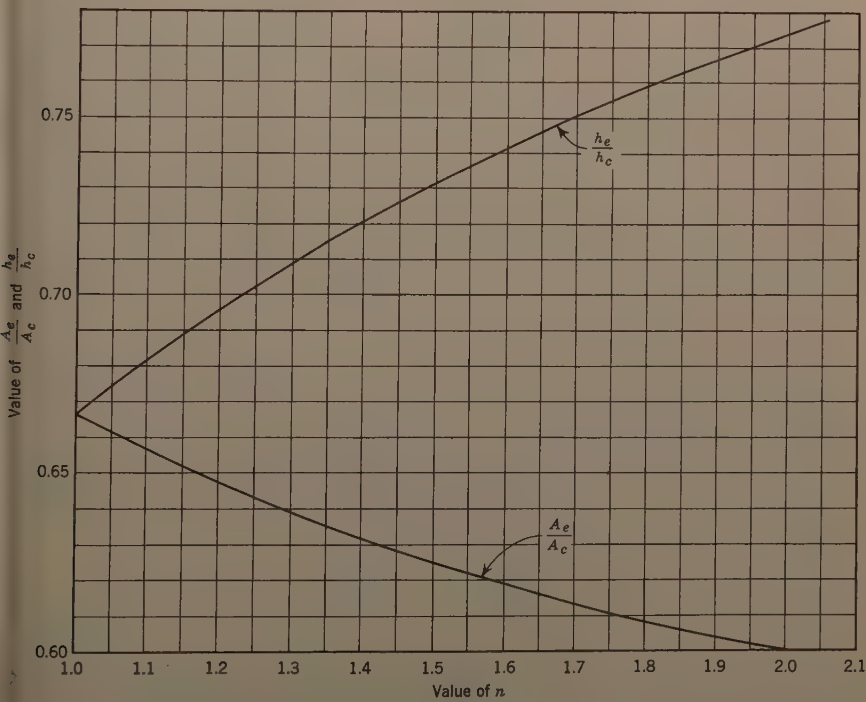

FIG. 3

the depth to the centroid of section (h_g). Using the dimensionless parameters, $X = m\ h/b$, the expressions obtained are

$$\bar{h} = \frac{1 + X}{1 + 2\ X}\ h \qquad (14)$$

and

$$h_g = \frac{3 + 2\ X}{6(1 + X)}\ h \qquad (15)$$

The ratio of the depth to centroid of area (h_g) to the mean depth $(\bar{h})$ is

$$\frac{h_g}{\bar{h}} = \frac{3 + 8X + 4X^2}{6\left(1 + 2X + X^2\right)} \qquad (16)$$

Introducing this relationship into Eq. 5b and using the notation $X_c = m\ h_c/b$ for the dimensionless factor X at the apparent critical section, the ratio of the area

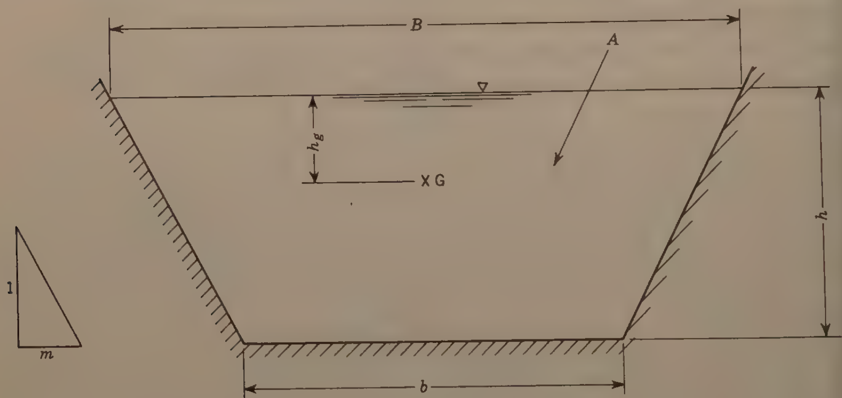

FIG. 4

of the end section (A_e) to the area of the apparent critical section (A_c) in trapezoidal channels becomes

$$\frac{A_e}{A_c} = \frac{6\left(1 + 2\ X_c + X_c^2\right)}{9 + 20\ X_c + 10\ X_c^2} \qquad (17)$$

or

$$\frac{A_e}{A_c} = 0.600 + \frac{0.600}{9 + 20\ X_c + 10\ X_c^2} \qquad (18)$$

Numerical values of the ratio (A_e/A_c) as a function of $X_c = m\ h_c/b$, compute by Eq. 18, are presented graphically in Fig. 5(a). The information is pre sented in a convenient form in Table 2 in which values of the ratio A_e/A_c ar

given as a function of h_c/b in a number of trapezoidal channels of various side slopes (m). The limits of the A_e/A_c ratio are: $A_e/A_c = 0.667$ for rectangular channels ($X_c = 0$), and $A_e/A_c = 0.600$ for triangular channels ($X_c = \infty$).

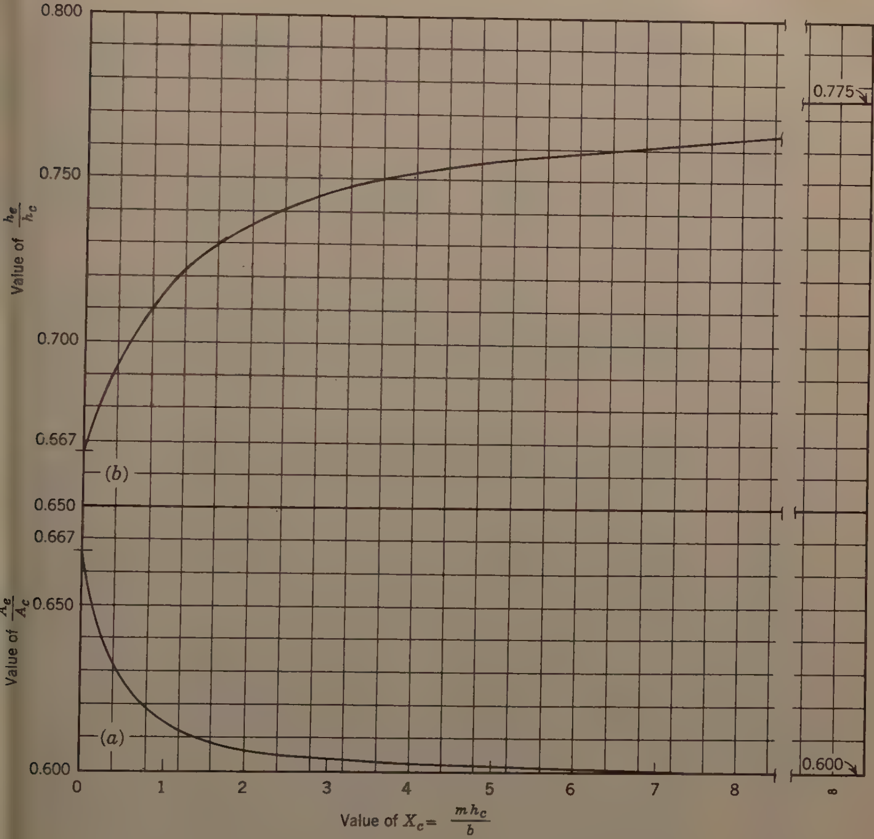

FIG. 5

An expression for the end depth in trapezoidal channels may be obtained by introducing into Eq. 17 the geometrical relationship

$$\frac{A_e}{A_c} = \frac{b\,h_e + m\,h_e^2}{b\,h_c + m\,h_c^2} = \frac{X_e + X_e^2}{X_c + X_c^2} \quad \ldots\ldots\ldots\ldots\ldots \quad (19)$$

in which $X_e = (m\,h_e/b)$.

Solving the equation, obtained by the preceding substitution, for X_e and simplifying, the following expression is obtained:

$$X_e = \frac{1}{2}\left[-1 + (1 + 4\,Z_c)^{1/2}\right] \quad \ldots\ldots\ldots\ldots\ldots \quad (20)$$

TABLE 2.—VALUES OF A_e/A_c

$\frac{h_c}{b}$	Values of m: $\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	4
0.05	0.6657	0.6649	0.6631	0.6614	0.6598	0.6569	0.6541	0.6515	0.6491	0.6448
0.10	0.6649	0.6631	0.6598	0.6569	0.6541	0.6491	0.6448	0.6410	0.6378	0.6323
0.15	0.6640	0.6614	0.6569	0.6528	0.6491	0.6428	0.6378	0.6335	0.6300	0.6244
0.20	0.6631	0.6598	0.6541	0.6491	0.6448	0.6378	0.6323	0.6279	0.6244	0.6191
0.25	0.6623	0.6583	0.6515	0.6458	0.6410	0.6335	0.6279	0.6236	0.6202	0.6154
0.30	0.6614	0.6569	0.6491	0.6428	0.6378	0.6300	0.6244	0.6202	0.6127	0.6127
0.35	0.6607	0.6554	0.6468	0.6402	0.6348	0.6270	0.6215	0.6178	0.6146	0.6106
0.40	0.6598	0.6541	0.6448	0.6378	0.6323	0.6244	0.6191	0.6154	0.6127	0.6090
0.45	0.6591	0.6528	0.6428	0.6355	0.6300	0.6222	0.6171	0.6136	0.6110	0.6077
0.50	0.6583	0.6515	0.6410	0.6335	0.6279	0.6202	0.6154	0.6121	0.6098	0.6067
0.6	0.6569	0.6491	0.6378	0.6300	0.6244	0.6171	0.6127	0.6098	0.6077	0.6052
0.7	0.6554	0.6468	0.6348	0.6270	0.6215	0.6146	0.6106	0.6081	0.6063	0.6042
0.8	0.6541	0.6448	0.6323	0.6244	0.6191	0.6127	0.6090	0.6067	0.6052	0.6034
0.9	0.6528	0.6428	0.6300	0.6222	0.6171	0.6110	0.6077	0.6057	0.6044	0.6028
1.0	0.6515	0.6410	0.6279	0.6202	0.6154	0.6098	0.6067	0.6049	0.6038	0.6024
1.1	0.6502	0.6393	0.6260	0.6186	0.6139	0.6087	0.6059	0.6043	0.6033	0.6021
1.2	0.6491	0.6378	0.6244	0.6171	0.6127	0.6077	0.6052	0.6038	0.6028	0.6018
1.3	0.6480	0.6362	0.6229	0.6158	0.6116	0.6070	0.6047	0.6034	0.6025	0.6016
1.4	0.6468	0.6348	0.6215	0.6146	0.6106	0.6063	0.6042	0.6030	0.6022	0.6014
1.5	0.6458	0.6335	0.6202	0.6136	0.6098	0.6057	0.6038	0.6027	0.6020	0.6013
1.6	0.6448	0.6323	0.6191	0.6127	0.6090	0.6052	0.6034	0.6024	0.6018	0.6011
1.7	0.6438	0.6311	0.6181	0.6118	0.6083	0.6048	0.6031	0.6022	0.6016	0.6010
1.8	0.6428	0.6300	0.6171	0.6110	0.6077	0.6044	0.6028	0.6020	0.6015	0.6009
1.9	0.6419	0.6289	0.6162	0.6104	0.6072	0.6041	0.6026	0.6018	0.6013	0.6008
2.0	0.6410	0.6279	0.6154	0.6098	0.6067	0.6038	0.6024	0.6017	0.6013	0.6007

in which

$$Z_c = \frac{6\,X_c(1 + X_c)^3}{9 + 20\,X_c + 10\,X_c^2} \qquad (21)$$

is a function of the apparent critical depth h_c. It is thus possible to compute the end depth in a given trapezoidal channel if the apparent critical depth is known.

A simpler method for obtaining the end depth in a trapezoidal channel is by a numerical solution. Starting with the known values of (A_e/A_c), for various given values of (h_c/b) from Table 2, values of the non-dimensional end area (A_e/b^2) are obtained by

$$\frac{A_e}{b^2} = \left(\frac{A_e}{A_c}\right)\left(\frac{A_c}{b^2}\right) \qquad (22)$$

in which (A_c/b^2) are cross-sectional areas corresponding to (h_c/b) values in the given channel. Values of (h_e/b) are then computed from (A_e/b^2) values by successive approximation using the relationship

$$\frac{A_e}{b^2} = \frac{h_e}{b} + m\left(\frac{h_e}{b}\right)^2 \qquad (23)$$

Finally the ratio (h_e/h_c) is computed from the values of (h_e/b) obtained for the various values of h_c/b in the given channel.

Values of (A_e/b^2), (h_e/b), and (h_e/h_c) computed by the preceding procedure for various (h_c/b) values in a number of trapezoidal channels are given in Tables 3, 4, and 5, respectively. The (h_e/h_c) values fall in the range of from $(h_e/h_c) = 0.667$ for rectangular channels to $h_e/h_c = 0.775$ for triangular channels. As the values in Table 5 were obtained by numerical solution, the values given may have an error of up to $\pm$ 0.002. The relationship between the ratio h_e/h_c and the critical depth factor $X_c = m\,h_c/b$ is shown graphically in Fig. 5(b). Although it describes the solution to Eqs. 20 and 21 the curve was actually plotted from values given in Table 5 with corresponding values of $m\,h_c/b$ computed from m and h_c/b values.

END DEPTH - DISCHARGE RELATIONSHIP

The relationship between the discharge (Q) and the apparent critical depth (h_c) is defined by Eq. 1, which could also be written as

$$Q = \sqrt{g}\,A_c\,\bar{h}_c^{\,1/2} \qquad (24)$$

As the ratio of the end depth to the apparent critical depth has a definite value for each value of the critical depth, it is possible to combine this ratio with Eq. 24 to obtain a direct relationship between the end depth and the discharge. This is useful for cases in which the end depth is used for estimation of discharge.

The relationship between the end depth and the discharge could be obtained directly in the case of exponential channels. In other cases it is simpler to present the relationship in a tabular form giving numerical values of end depth and of discharge for the same values of apparent critical depth.

TABLE 3.—VALUES OF A_e/b^2

$\frac{h_c}{b}$	Values of m									
	m	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	4
0.05	0.0335	0.0336	0.0339	0.0343	0.0346	0.0353	0.0359	0.0366	0.0373	0.0387
0.10	0.0673	0.0680	0.0693	0.0706	0.0720	0.0746	0.0774	0.0801	0.0829	0.0885
0.15	0.1015	0.1029	0.1059	0.1090	0.1120	0.1181	0.1244	0.1306	0.1370	0.1500
0.20	0.1359	0.1386	0.1439	0.1493	0.1548	0.1658	0.1770	0.1883	0.1998	0.2229
0.25	0.1707	0.1748	0.1832	0.1917	0.2003	0.2178	0.2355	0.2533	0.2713	0.3077
0.30	0.2058	0.2119	0.2239	0.2362	0.2487	0.2740	0.2997	0.3256	0.3517	0.4044
0.35	0.2414	0.2494	0.2660	0.2829	0.2999	0.3347	0.3698	0.4054	0.4410	0.5129
0.40	0.2771	0.2878	0.3095	0.3316	0.3541	0.3996	0.4458	0.4923	0.5392	0.6334
0.45	0.3133	0.3268	0.3543	0.3825	0.4111	0.4690	0.5276	0.5867	0.6462	0.7657
0.50	0.3497	0.3664	0.4006	0.4355	0.4709	0.5427	0.6154	0.6886	0.7622	0.9100
0.6	0.4237	0.4479	0.4975	0.5481	0.5994	0.7035	0.8088	0.9147	1.0209	1.2346
0.7	0.4989	0.5320	0.5999	0.6693	0.7396	0.8820	1.0258	1.1706	1.3157	1.6072
0.8	0.5756	0.6190	0.7082	0.7992	0.8915	1.0784	1.2667	1.4561	1.6461	2.0274
0.9	0.6536	0.7087	0.8222	0.9380	1.0552	1.2923	1.5314	1.7717	2.0126	2.4956
1.0	0.7329	0.8012	0.9418	1.0854	1.2308	1.5245	1.8201	2.1172	2.4152	3.0120
1.1	0.8135	0.8966	1.0673	1.2418	1.4181	1.7744	2.1328	2.4927	2.8536	3.5765
1.2	0.8958	0.9950	1.1988	1.4070	1.6175	2.0419	2.4692	2.8982	3.3275	4.1885
1.3	0.9793	1.0959	1.3361	1.5811	1.8287	2.3278	2.8300	3.3338	3.8379	4.8489
1.4	1.0640	1.1998	1.4792	1.7639	2.0516	2.6313	3.2143	3.7989	4.3840	5.5569
1.5	1.1503	1.3066	1.6280	1.9558	2.2868	2.9528	3.6228	4.2942	4.9665	6.3136
1.6	1.2380	1.4164	1.7830	2.1567	2.5334	3.2923	4.0548	4.8192	5.5847	7.1170
1.7	1.3270	1.5288	1.9439	2.3661	2.7921	3.6500	4.5112	5.3746	6.2386	7.9693
1.8	1.4173	1.6443	2.1105	2.5845	3.0628	4.0253	4.9912	5.9598	6.9293	8.8693
1.9	1.5092	1.7625	2.2830	2.8124	3.3457	4.4190	5.4957	6.5746	7.6545	9.8171
2.0	1.6025	1.8837	2.4616	3.0490	3.6402	4.8304	6.0240	7.2204	8.4182	10.8126

TABLE 4.—VALUES OF h_e/b

$\frac{h_c}{b}$	Values of m									
	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	4
0.05	0.0334	0.0334	0.0334	0.0335	0.0335	0.0336	0.0337	0.0338	0.0339	0.0341
0.10	0.0668	0.0669	0.0671	0.0672	0.0675	0.0677	0.0681	0.0684	0.0687	0.0693
0.15	0.1002	0.1004	0.1008	0.1013	0.1017	0.1024	0.1031	0.1037	0.1044	0.1055
0.20	0.1337	0.1341	0.1348	0.1355	0.1362	0.1375	0.1386	0.1396	0.1405	0.1421
0.25	0.1672	0.1678	0.1689	0.1700	0.1711	0.1730	0.1745	0.1759	0.1772	0.1792
0.30	0.201	0.202	0.203	0.205	0.206	0.209	0.211	0.213	0.214	0.217
0.35	0.235	0.236	0.238	0.240	0.242	0.245	0.247	0.250	0.251	0.254
0.40	0.268	0.270	0.272	0.275	0.277	0.281	0.284	0.287	0.289	0.292
0.45	0.302	0.304	0.307	0.310	0.313	0.318	0.321	0.324	0.326	0.330
0.50	0.336	0.338	0.342	0.346	0.349	0.354	0.358	0.362	0.364	0.368
0.6	0.403	0.407	0.412	0.417	0.422	0.428	0.433	0.437	0.440	0.444
0.7	0.471	0.475	0.483	0.490	0.495	0.503	0.508	0.513	0.517	0.521
0.8	0.539	0.545	0.554	0.562	0.568	0.578	0.584	0.589	0.593	0.598
0.9	0.608	0.614	0.626	0.635	0.642	0.653	0.660	0.665	0.669	0.675
1.0	0.676	0.684	0.698	0.709	0.717	0.728	0.736	0.742	0.746	0.752
1.1	0.744	0.754	0.771	0.783	0.792	0.804	0.813	0.818	0.823	0.829
1.2	0.813	0.825	0.843	0.857	0.866	0.880	0.889	0.895	0.900	0.906
1.3	0.882	0.895	0.916	0.931	0.942	0.956	0.966	0.972	0.977	0.983
1.4	0.951	0.966	0.990	1.006	1.017	1.032	1.042	1.049	1.054	1.060
1.5	1.020	1.038	1.063	1.080	1.093	1.109	1.119	1.126	1.131	1.138
1.6	1.090	1.109	1.137	1.155	1.168	1.185	1.196	1.203	1.208	1.215
1.7	1.159	1.180	1.211	1.231	1.244	1.262	1.272	1.280	1.285	1.292
1.8	1.229	1.252	1.285	1.306	1.320	1.338	1.351	1.357	1.362	1.369
1.9	1.298	1.324	1.359	1.381	1.396	1.415	1.426	1.434	1.439	1.447
2.0	1.368	1.396	1.434	1.457	1.472	1.492	1.503	1.511	1.517	1.524

TABLE 5.—VALUES OF h_e/h_c

$\frac{h_c}{b}$	Values of m									
	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	4
0.05	0.668	0.668	0.668	0.670	0.670	0.672	0.674	0.676	0.678	0.682
0.10	0.668	0.669	0.671	0.672	0.675	0.677	0.681	0.684	0.687	0.693
0.15	0.668	0.669	0.672	0.675	0.678	0.687	0.687	0.690	0.696	0.703
0.20	0.668	0.670	0.674	0.678	0.681	0.688	0.693	0.698	0.703	0.710
0.25	0.669	0.671	0.676	0.680	0.684	0.692	0.698	0.704	0.709	0.717
0.30	0.670	0.673	0.677	0.683	0.687	0.697	0.701	0.710	0.713	0.723
0.35	0.671	0.674	0.680	0.686	0.692	0.700	0.706	0.714	0.717	0.726
0.40	0.670	0.675	0.680	0.687	0.694	0.702	0.710	0.717	0.722	0.730
0.45	0.671	0.676	0.682	0.689	0.696	0.707	0.712	0.720	0.725	0.733
0.50	0.672	0.676	0.684	0.692	0.698	0.708	0.716	0.724	0.728	0.736
0.6	0.672	0.678	0.687	0.695	0.703	0.713	0.722	0.728	0.733	0.740
0.7	0.673	0.679	0.690	0.700	0.707	0.719	0.726	0.733	0.739	0.744
0.8	0.674	0.681	0.692	0.702	0.710	0.722	0.730	0.736	0.741	0.748
0.9	0.675	0.682	0.696	0.706	0.713	0.726	0.733	0.739	0.744	0.750
1.0	0.676	0.684	0.698	0.709	0.717	0.728	0.736	0.742	0.746	0.752
1.1	0.676	0.686	0.701	0.712	0.720	0.731	0.739	0.744	0.748	0.754
1.2	0.677	0.687	0.703	0.714	0.722	0.733	0.741	0.746	0.750	0.755
1.3	0.678	0.688	0.705	0.716	0.725	0.735	0.743	0.748	0.752	0.756
1.4	0.679	0.690	0.707	0.718	0.726	0.737	0.744	0.749	0.753	0.757
1.5	0.680	0.692	0.709	0.720	0.729	0.739	0.746	0.751	0.754	0.759
1.6	0.681	0.693	0.711	0.722	0.730	0.742	0.748	0.752	0.755	0.759
1.7	0.682	0.694	0.712	0.724	0.732	0.742	0.749	0.753	0.756	0.760
1.8	0.683	0.696	0.714	0.726	0.733	0.743	0.750	0.754	0.757	0.761
1.9	0.683	0.697	0.715	0.727	0.735	0.745	0.751	0.755	0.757	0.762
2.0	0.684	0.698	0.717	0.728	0.736	0.746	0.752	0.756	0.758	0.762

TABLE 6.—VALUES OF DISCHARGE (FROM EQ. 29)

$\frac{h_c}{b}$	Values of m: 0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	4
0.05	0.0112	0.0112	0.0112	0.0113	0.0114	0.0115	0.0116	0.0118	0.0119	0.0121	0.0124
0.10	0.0316	0.0318	0.0320	0.0324	0.0329	0.0333	0.0342	0.0351	0.0361	0.0372	0.0390
0.15	0.0581	0.0586	0.0592	0.0604	0.0616	0.0628	0.0655	0.0681	0.0709	0.0734	0.0793
0.20	0.0894	0.0906	0.0917	0.0942	0.0967	0.0994	0.1048	0.1105	0.1162	0.1220	0.1340
0.25	0.1250	0.1270	0.1290	0.1334	0.1380	0.1426	0.1524	0.1624	0.1725	0.1830	0.2042
0.30	0.1643	0.1675	0.1708	0.1777	0.1850	0.1926	0.2081	0.2242	0.2406	0.2572	0.2908
0.35	0.2071	0.2117	0.2166	0.2270	0.2379	0.2491	0.2725	0.2962	0.3206	0.3452	0.3950
0.40	0.2530	0.2595	0.2664	0.2811	0.2965	0.3124	0.3452	0.3789	0.4131	0.4477	0.5175
0.45	0.3019	0.3108	0.3200	0.3398	0.3608	0.3824	0.4270	0.4726	0.5186	0.5653	0.6595
0.50	0.3536	0.3650	0.3773	0.4035	0.4309	0.4593	0.5176	0.5774	0.6379	0.6988	0.8216
0.6	0.4647	0.4830	0.5027	0.5446	0.5887	0.6341	0.7274	0.8225	0.9186	1.0154	1.2099
0.7	0.5857	0.6127	0.6420	0.7046	0.7703	0.8379	0.9762	1.1170	1.2589	1.4018	1.6886
0.8	0.7155	0.7536	0.7950	0.8835	0.9764	1.0715	1.2661	1.4637	1.6627	1.8627	2.2643
0.9	0.8521	0.9051	0.9614	1.0816	1.2073	1.3364	1.5989	1.8653	2.1329	2.4019	2.9415
1.0	1.0000	1.0673	1.1411	1.2990	1.4642	1.6330	1.9765	2.3238	2.6729	3.0236	3.7270
1.1	1.154	1.239	1.334	1.536	1.747	1.963	2.400	2.842	3.286	3.731	4.625
1.2	1.314	1.422	1.540	1.794	2.057	2.326	2.872	3.422	3.974	4.529	5.640
1.3	1.482	1.614	1.760	2.071	2.395	2.725	3.393	4.066	4.742	5.420	6.777
1.4	1.656	1.816	1.993	2.370	2.762	3.159	3.965	4.776	5.591	6.406	8.042
1.5	1.837	2.010	2.182	2.690	3.157	3.631	4.590	5.555	6.523	7.493	9.436
1.6	2.024	2.248	2.499	3.031	3.580	4.140	5.269	6.404	7.754	8.682	10.967
1.7	2.216	2.479	2.772	3.394	4.037	4.688	6.002	7.325	8.880	9.978	12.637
1.8	2.415	2.719	3.059	3.780	4.523	5.276	6.793	8.321	10.098	11.383	14.450
1.9	2.619	2.968	3.360	4.188	5.041	5.903	7.644	9.392	11.143	12.898	16.410
2.0	2.828	3.228	3.674	4.619	5.590	6.572	8.552	10.541	12.534	14.528	18.522

The relationship for exponential channels is obtained by introducing Eqs. 6 and 7 into Eq. 24;

$$Q = \frac{\sqrt{g}\,K}{\sqrt{n}}\, h_c^{\frac{2n+1}{2}} \quad \text{. . . . . (25)}$$

Substituting into this equation the expression for h_e/h_c in exponential channels (Eq. 10), the following relationship is obtained between the discharge and the end depth:

$$Q = \frac{\sqrt{g}\,K}{\sqrt{n}} \left(\frac{2n+1}{n+1}\right)^{\frac{2n+1}{2\,n}} h_e^{\frac{2n+1}{2}} \quad \text{. . . . . (26)}$$

Thus for rectangular channels $(n = 1.0;\ K = B)$

$$Q = \sqrt{g}\ B\,(3/2)^{3/2}\, h_e^{3/2} \text{. . . . . (27a)}$$

or

$$Q = 1.837\ \sqrt{g}\ B\, h_e^{3/2} \text{ . . . . (27b)}$$

and for triangular channels $(n = 2.0;\ K = m)$

$$Q = \frac{\sqrt{g}\,m}{\sqrt{2}} \left(\frac{5}{3}\right)^{5/4} h_e^{5/2} \text{ . . . . (28a)}$$

or

$$Q = 1.339\ \sqrt{g}\ m\, h_e^{5/2} \text{ . . . . (28b)}$$

The relationship between discharge and end depth in trapezoidal channels is obtainable by combining values of discharge computed from

$$\frac{Q}{(g\,b^5)^{1/2}} = \left(\frac{A_c}{b^2}\right)\left(\frac{\bar{h}_c}{b}\right)^{1/2} \text{ . . . . (29)}$$

with values of (h_e/b) corresponding to the same critical depths. Values of the discharge computed by Eq. 29 are given in Table 6 and values of the end depth are given in Table 4 for the same parameters (m and h_c/b).

EXPERIMENTAL RESULTS

As mentioned previously, the end depth in rectangular channels has been fairly extensively investigated in a number of places. In an analysis of experimental results available up to 1957, the writer[15] came to the conclusion that the measured values of the ratio (h_e/h_c) in rectangular channels fall in most cases in the range

$$0.67 < h_e/h_c < 0.73$$

and that the most probable value of this ratio as measured is $h_e/h_c = 0.70$ compared with the theoretical value $h_e/h_c = 0.67$, derived by the momentum equation. It was also concluded that the discrepancy could be explained by residual pressure in the end section, assumed to have zero pressure in the derivation of the momentum equation.

TABLE 7.—EXPERIMENTAL CHANNEL NO. I

Experiment Number	Station Number 0	1	2	3	4	5	6	Discharge, Q, in cubic feet per second	Apparent critical depth, h_c, in feet
	Distance from end section, L, in feet: 0	0.69	2.07	3.93	6.17	9.45	13.12		
	Depth at Station, h, in feet								
1	0.338	0.433	0.486	0.492	0.509	0.518	0.532	1.712	0.454
2	0.325	0.420	0.469	0.482	0.492	0.495	0.512	1.580	0.436
3	0.312	0.407	0.453	0.466	0.476	0.486	0.495	1.495	0.424
4	0.295	0.390	0.427	0.440	0.453	0.459	0.469	1.330	0.400
5	0.272	0.358	0.394	0.410	0.420	0.427	0.433	1.137	0.368
6	0.282	0.374	0.413	0.427	0.436	0.446	0.453	1.242	0.387
7	0.272	0.358	0.394	0.410	0.420	0.427	0.433	1.129	0.367
8	0.256	0.338	0.377	0.384	0.394	0.407	0.413	1.001	0.343
9	0.243	0.322	0.358	0.367	0.374	0.387	0.394	0.906	0.325
10	0.210	0.289	0.315	0.325	0.335	0.345	0.348	0.708	0.286
11	0.223	0.302	0.331	0.341	0.354	0.358	0.367	0.791	0.304
12	0.203	0.279	0.308	0.318	0.325	0.331	0.341	0.671	0.278
13	0.190	0.262	0.292	0.299	0.308	0.312	0.322	0.593	0.260
14	0.167	0.233	0.256	0.266	0.272	0.282	0.289	0.467	0.227
15	0.131	0.190	0.207	0.213	0.223	0.230	0.236	0.308	0.178
16	0.148	0.207	0.230	0.236	0.246	0.253	0.256	0.374	0.199
17	0.345	0.440	0.489	0.499	0.515	0.525	0.538	1.773	0.463
18	0.318	0.407	0.449	0.466	0.476	0.482	0.495	1.477	0.421

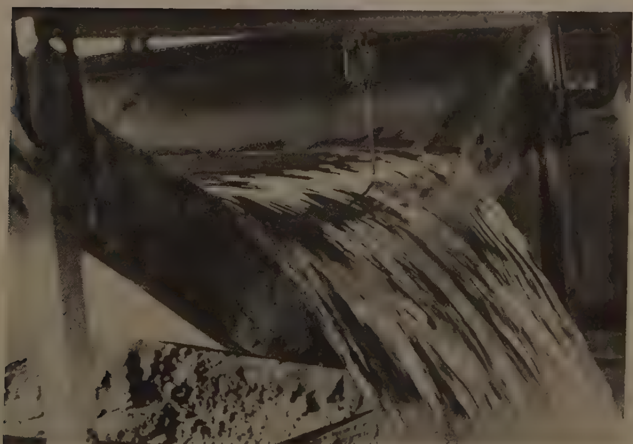

FIG. 6.—END SECTION IN CHANNEL NO. I

Experiments in trapezoidal channels were carried out on two channels which were part of existing models. The cross sections of the two channels investigated can be described by the following quantities:

channel No. I : b = 0.547 ft ; m = 1.5

channel No. II : b = 0.410 ft ; m = 2.0

The first channel was fairly long with a number of curves in plan. It received its water from a stilling tank. The channel was made of concrete and

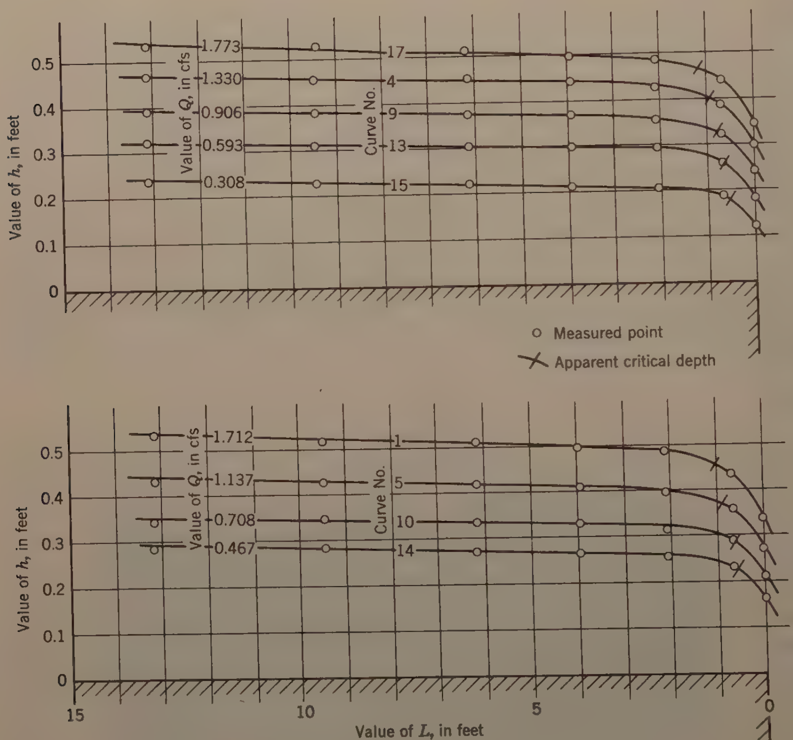

FIG. 7.—WATER SURFACE CURVES

finished accurately with smooth cement mortar. The end section of the channel was preceded by a curved portion about 12.1 ft long on a radius of 21.8 f The curved portion was preceded by a straight portion 8.7 ft long and befor this by another curved portion. The longitudinal slope of the channel was approximately 0.15 ft per 100 ft. The drop at the end was large enough for th nappe to be free at all discharges. The discharge was measured by a standar 90° triangular weir after the drop, equipped with a proper stilling chambe and read by a point gage with vernier to 0.3×10^{-3} ft.

Experimental channel No. II was a straight channel about 16 ft long, receiving its water from a stilling tank. The channel was made of steel sheets, painted with a special paint mixed with sand to increase the roughness of the channel, except for the last 2 ft-long portion preceding the end section. This last section was made of concrete and finished with cement mortar. The longitudinal slope of the channel was approximately 1.5 ft per 100 ft. The drop at the end was only 0.410 ft high and was followed by another trapezoidal channel of the same section. The low drop resulted in partial submergence of the lower part of the nappe at flows above 0.95 cfs. The discharge was measured at the inlet by a 10 in.-by-5 in. Venturi meter, with a water-air differential manometer, calibrated volumetrically before the experiments were started.

TABLE 8.—EXPERIMENTAL CHANNEL NO. II

Experiment Number (1)	Discharge, Q, in cubic feet per second (2)	End Depth, h_e, in feet (3)
1	1.692	0.328
2	1.558	0.318
3	1.452	0.302
4	1.261	0.285
5	0.996	0.249
6	0.872	0.227
7	0.668	0.203
8	0.579	0.194
9	0.766	0.217
10	0.957	0.243
11	1.053	0.259
12	1.095	0.266
13	1.166	0.272
14	1.363	0.295
15	1.519	0.312
16	1.621	0.322
17	0.940	0.243
18	1.381	0.302
19	1.491	0.308
20	1.467	0.328

The water supply for the two channels came from the constant head water tower of the laboratory and enough time was allowed in all cases for conditions to become steady.

The end depths in the two channels, and in channel No. I as well as depths along the channel, were measured with point gages reading to 3×10^{-3} ft. Readings in all cases were taken only on the axis of the channels. A check on readings at the water surface at other points of the same cross sections indicated that the differences were of the same order as the limit of reading. Small fluctuations in the water surface were present at almost all flows. The reading was taken after adjusting the gage so that its point was approximately equal periods of time in and out of the water. The nature of the water surface at the fall can be seen in Fig. 6.

The results of water surface measurements in channel No. I are given in Table 7 and typical water surface curves are plotted in Fig. 7. The apparent

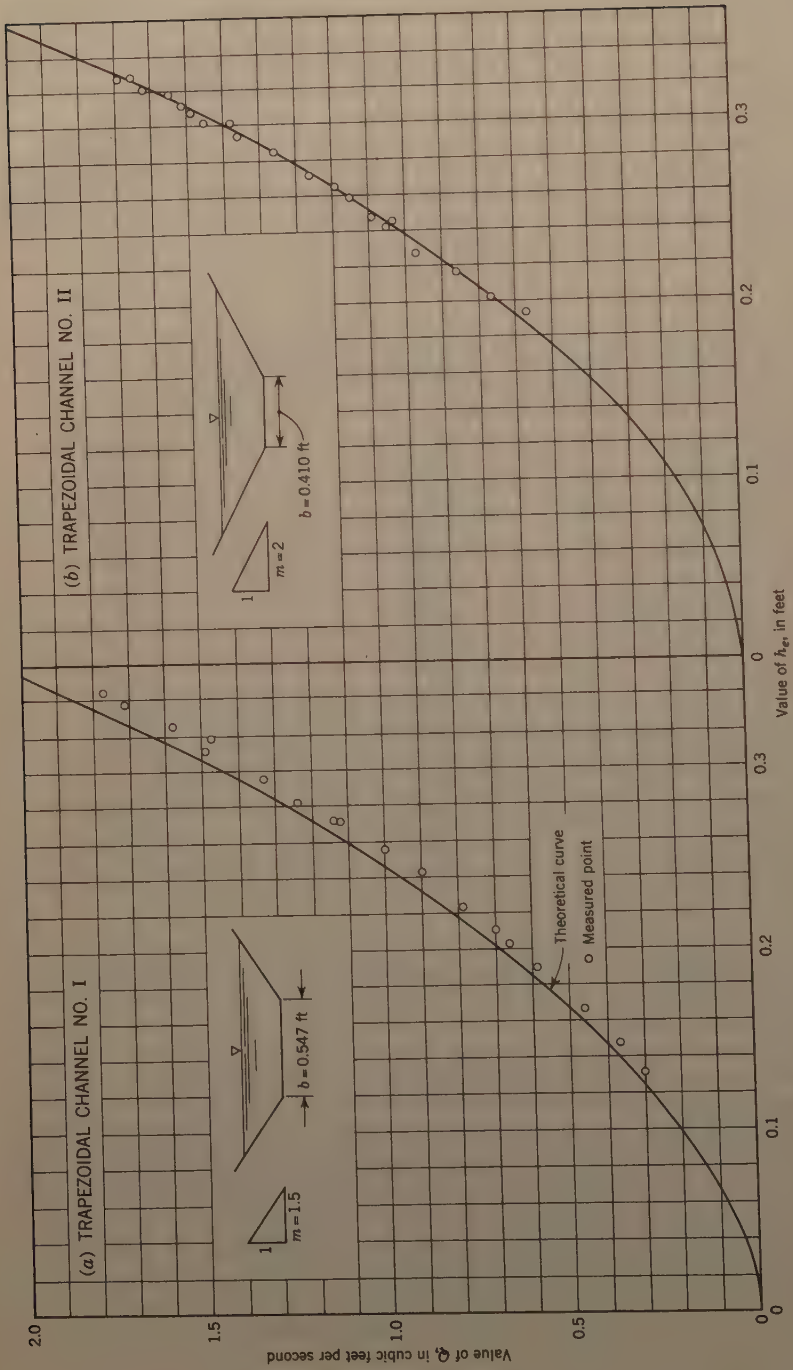

FIG. 8.—END DEPTH DISCHARGE

critical depth in Table 7 was computed from the measured discharge with the aid of Table 6. The location of the apparent critical section is marked on the water surface curves in Fig. 7 and it is seen that it falls between 2 h_c and 3 h_c back from the end section. A plot of end depth as a function of the discharge is given in Fig. 8(a) and is compared with the theoretical curve derived from Tables 4 and 6.

The results of end depth measurements in channel No. II are given in Table 8. A plot of the end depth as function of the discharge is given in Fig. 8(b) and is compared with the theoretical curves derived from Tables 4 and 6.

The comparison of the measured and computed end depths, Figs. 8(a) and 8(b), shows fairly good agreement. The values measured in channel No. II are scattered about the theoretical curve with deviations of about $\pm$ 2%, while for channel No. I the measured values of the end depth are all higher than the theoretical curve deviating from it by + 2% to + 5%.

The different behavior of the two channels is probably due to one or more of their different physical features, straightness, slope, and roughness influencing the pressure distribution in the end section. An end depth higher than the theoretical value indicates that the residual pressure in the end section was of such a magnitude that it could not be neglected. A detailed examination of the effect of the various factors mentioned previously on the pressure distribution at the end section was not included in the investigation reported herein.

CONCLUSIONS

Equations relating the depth, at the end section before a vertical drop in trapezoidal channels on mild slopes, to the apparent critical depth and hence to the discharge, may be derived by using the momentum equation.

Experimental results, obtained in two model trapezoidal channels, agree fairly well with the theoretical relationships derived. While more experimental results are desired, it may be concluded at this stage that the results based on the momentum equation are adequate for the purpose of designs of drop structures. The results are not sufficiently accurate to be used for flow measurements, without prior calibration, unless some additional research on the factors affecting the deviations of values of the end depth from the theoretical values is carried out.

ACKNOWLEDGMENT

The experimental work reported on herein was performed at the Hydraulic Laboratory of the Technion, Israel Institute of Technology, Haifa, Israel.

APPENDIX.—NOTATION

The following symbols, adopted for use in this paper, conform essentially with "American Standard Letter Symbols for Hydraulics" (ASA Z10.2-1942),

prepared by a committee of the American Standards Association with Societ
representation, and approved by the Association in 1942:

A = cross-sectional area of flow;

B = width of channel at water surface;

b = bottom width of trapezoidal channel;

c = subscript denoting quantity related to apparent critical section;

e = subscript denoting quantity related to end section;

g = acceleration due to gravity;

h = depth of water in channel;

h_g = depth of centroid of section below water surface;

$\overline{h}$ = mean depth of section: $\overline{h} = A/B$;

h_c = apparent critical depth defined by the relationship: $Q^2 B_c/g A_c^3 = 1$

h_e = end depth, depth of water at end section just before the drop;

h_n = depth of uniform flow in channel;

K = coefficient in equation for area in exponential channels;

L = distance upstream from end section;

L_c = distance from end section to apparent critical section;

m = side slope of channel, horizontal to vertical;

n = exponent in equation for area in exponential channels;

Q = discharge;

S = longitudinal slope of channel;

S_c = critical slope of channel;

V = mean velocity in section: $V = Q/A$;

X = dimensionless factor: $X = m\,h/b$;

Z = dimensionless factor : $A = 6\,S(1 + X)^3/(9 + 20\,X + 10\,X^2)$;

β = momentum transport coefficient; and

γ = specific weight of liquid.

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SYNTHETIC UNIT HYDROGRAPHS FOR SMALL WATERSHEDS

By Don M. Gray[1]

SYNOPSIS

To apply the unit-graph principle on an ungaged area, a representative unit graph for the watershed must be selected. A method is presented herein whereby the unit graphs for small watersheds can be synthesized from measurable topographic characteristics. In the development of the procedure, a two-parameter equation was used to describe the hydrograph. A successful linkage was established to permit the evaluation of these parameters from measurements of the length and slope of the main stream as taken from topographic maps.

INTRODUCTION

An approximation of the discharge hydrograph for a given watershed resulting from a given rainfall pattern can be obtained by application of the infiltration theory and the unit-hydrograph principle. The reliability of this approximation is limited in part by the success with which the unit hydrograph for the watershed can be derived.

Several methods can be used to develop the unit hydrograph for an ungaged area from measurable physical characteristics. In general, however, most of these methods use empirical relationship obtained either from large watersheds (10 sq miles to 10,000 sq miles) or from small watersheds (less than 1

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sq mile). Relatively few studies have been conducted on watersheds within the intermediate size group.

Hydrologic investigations on watersheds of intermediate size are not without importance, however. A large percentage of the hydraulic structures constructed by the Soil Conservation Service and Bureau of Public Roads are designed to handle surface runoff originating from watersheds less than 25 sq miles to 50 sq miles in size. Additional hydrologic investigations on these watersheds are desirable because of the large expenditures invested annually for facilities used in the control and conservation of runoff from these areas and the relatively inadequate data on which the design of these facilities is based.

This paper describes a procedure whereby the unit hydrograph can be synthesized for small watershed areas. It presents the methodology and necessary relationships to perform this approximation once pertinent characteristics of the watershed are known.

THEORY

In 1932 L. K. Sherman, F. ASCE, (16)[2] presented the concept of the unit graph. A unit graph is a discharge hydrograph representing 1 in. of direct runoff generated uniformly over the tributary area at a uniform rate during a specified period of time. It represents the integrated effects of all the sensibly constant physical characteristics of a watershed on the translation and storage of a given rainfall excess as it flows from the watershed. Thus, pertinent features of the unit graph would be expected to be related to the physical characteristics of the area.

Among the recent contributions to the field of hydrology has been the development of theoretical expressions to define the geometry of the instantaneous unit graph (unit graph resulting from a rainfall excess of 1 in. generated during zero time). Two mathematical expressions have been proposed; one by C. G. Edson, M. ASCE, (6), and the other by J. E. Nash (14). These are given by Eqs. 1a and 2, respectively.

Edson.—

$$Q_t = \frac{V_y}{\Gamma(z+1)} (y\,t)^z\, e^{-yt} \quad \ldots\ldots\ldots\ldots \quad (1a)$$

in which Q_t denotes instantaneous discharge rate at time, t; V is the volume of surface runoff; y represents the recession constant whose magnitude is greater than zero as determined from the slope of the recession curve plotted on semi-logarithmic paper; z is the exponent whose magnitude depends on the shape of the time-area concentration curve of the watershed; e is the base of the natural logarithms; and Γ denotes the gamma function.

By setting z = m-1 and simplifying, Eq. 1a can be written in a more useful form

$$Q_t = \frac{V\,y^m}{\Gamma(m)}\, e^{-yt}\, t^{m-1} \quad \ldots\ldots\ldots\ldots \quad (1b)$$

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2 Numerals in parentheses refer to corresponding items in the Appendix-Selected Bibliography.

Nash.—

$$Q_t = \frac{V\,k^{-n}}{\Gamma(n)}\,e^{-t/k}\,t^{n-1} \quad \ldots\ldots\ldots\ldots\ (2)$$

in which Q_t, V, Γ, e and t connotate the same definition as implied previously; n is the shape parameter whose magnitude depends on the storage properties of the watershed; and k is the storage constant. As shown by Eqs. 1 and 2, by evaluating 2 parameters, the complete unit hydrograph can be described.

The relationships serve as a useful tool in developing a rational synthetic procedure. They provide the investigator the opportunity to seek a solution in logical sequence from reason to result. In addition, the use of point correlations as used almost exclusively in the past to relate the physical characteristics of the watershed with properties of the unit graph can be eliminated. Edson (6) suggests that the general failure encountered in correlating basin characteristics and the hydrograph properties, peak discharge and period of rise, may be attributed to the complex relation of y and z (Eq. 1a) so as to restrict a satisfactory tie-in. Another important advantage for describing the unit hydrograph mathematically is the adaptability of the equational form for use in high speed computers.

Data.—Hydrologic and topographic data were obtained from 42 watersheds located in Illinois, Iowa, Missouri, Nebraska, Ohio, and Wisconsin. A list of the pertinent hydrologic and topographic data from these watersheds that are used in this paper is given in Table 1. The watershed characteristics are shown in Fig. 1.

Selection of Hydrologic Data.—In the study, the unit-storm concept proposed by C. O. Wisler, F. ASCE, and E. F. Brater, F. ASCE, (22) was accepted. A unit storm is a storm of such duration that the period of surface runoff is not appreciably less for any storm of shorter duration. The duration of the unit storm varies with watershed size. For small watersheds it approaches the period of rise of the unit graph; for large watersheds, it may be only a fraction of that time.

B. S. Barnes, F. ASCE, (1), M. M. Bernard (2), and Brater (3) have suggested several criteria to be followed in selecting hydrologic data suitable for distribution-graph and/or unit-hydrograph development. These were summarized to formulate the basis of the following list of standards used in this study.

1. The rain must have fallen within the selected time unit and must not have extended beyond the period of rise of the hydrograph.
2. The storm must have been well distributed over the watershed, all stations showing an appreciable amount.
3. The storm period must have occupied a place of comparative isolation in the record.
4. The runoff following a storm must have been uninterrupted by the effects of low temperatures and unaccompanied by melting snow or ice.
5. The stage graphs or hydrographs must have a sharp, defined, rising limb culminating to a single peak and followed by an uninterrupted recession.
6. All stage graphs or hydrographs for the same watershed must show approximately the same period of rise.

TABLE 1.—TOPOGRAPHIC AND HYDROLOGIC DATA

State (1)	Number (2)	Watershed (3)	Topographic Properties: A, in sq miles[a] (4)	L, in miles[b] (5)	S_c, in %[c] (6)	Dimensionless Graph Properties: T_p, in minutes (7)	Peak ordinate, in % flow/$0.25T_p$ (8)
Illinois	1	W IV, Edwardsville	0.45	0.54	1.10	27	16.4
Iowa	2	Davids Creek near Hamlin	26.01	9.14	0.39	115	18.5
	3	Haywarth Main Outlet near Climbing Hill	0.91	1.80	1.41	15	12.4
	4	Indian Creek at Council Bluffs	7.36	5.65	0.49	40	16.3
	5	Mackey Creek near Mapleton	0.65	0.83	1.34	21	20.5
	6	Heppner Main Outlet near Mapleton	0.35	0.75	2.56	19	21.5
	7	Ralston Creek near Iowa City	3.00	3.50	0.45	79	21.3
	8	Rapid Creek near Iowa City	24.57	9.50	0.21	158	15.2
	9	Henneker Main Outlet near Anthon	0.89	1.78	0.54	18	21.0
	10	Waubonsie Creek near Bartlett	32.64	12.50	0.40	130	21.9
Missouri	11	Beaver Creek near Rolla	13.70	5.95	0.70	65	16.6
	12	Behnke Branch near Rolla	1.03	1.95	1.37	45	31.4
	13	Big Creek near Yukon	8.36	2.45	2.78	58	18.5
	14	Bourbeuse Creek near St. James	21.30	6.00	0.41	90	19.2
	15	Coyle Branch at Houston	1.30	1.21	0.[illegible]2	57	25.2
	16	East Fork Fishing River at Excelsior Springs	26.00	7.80	0.30	121	18.4
	17	Green Acre Branch near Rolla	0.62	0.98	1.45	30	25.9
	18	Jenkins Branch at Gower	2.72	2.50	0.53	90	20.2
	19	Lanes Fork near Rolla	0.23			53	15.1
	20	Lanes Fork near Vichy	24.10	9.40	0.40	87	20.7
	21	Little Beaver Creek near Rolla	6.27	3.10	1.02	60	21.0
	22	Lost Creek at Elsberry	12.20	3.70	0.74	75	21.8
	23	Mill Creek at Oregon	4.90	3.00	0.79	50	15.7
	24	Oak Grove Branch near Brighton	1.00	1.00		71	22.4
	25	Shiloh Branch near Marshall	2.37	2.45	0.45	61	16.4
	26	Stahl Creek near Miller	3.86	2.70	0.55	128	25.0
	27	Stark's Creek at Preston	4.72	1.58		97	18.2
	[illegible]	[illegible]	[illegible]	4.60	0.28	238	27.2

Nebraska	29	Dry Creek near Curtis	20.00	11.59	0.30	150	28.8
	30	W-3, Hastings	0.75	1.96	0.86	55	22.0
	31	New York Creek near Herman	30.00	10.25	0.26	198	24.2
	32	Tekamah Creek at Tekamah	21.53	7.50	0.52	180	35.8
Ohio	33	W-5, Coshocton	0.55	0.92	2.64	32	23.1
	34	W-11, Coshocton	0.46	1.17	1.83	30	18.3
	35	W-91, Coshocton	0.46	1.31	2.13	27	21.2
	36	W-92, Coshocton	1.44	1.56	1.84	55	19.4
	37	W-94, Coshocton	2.37	2.41	1.37	45	16.4
	38	W-95, Coshocton	4.02	3.25	1.09	75	18.0
	39	W-97, Coshocton	7.16	5.11	0.72	99	19.7
	40	W-196, Coshocton	0.47	0.88	3.94	14	14.8
Wisconsin	41	W-I, Fennimore	0.52	0.99	1.80	18	17.2
	42	W-II, Fennimore	0.27	0.46	2.20	19	16.7

[a] Drainage-area size, A - plane area of the watershed in square miles which is enclosed within the topographic divide above the gaging station.

[b] Length of main stream, L - distance in miles along the main stream from the gaging station to the outermost point on the stream channel as defined on the topographic map (Fig. 1). The main stream is the stream of highest order that passes through the gaging station. To delineate the main stream at bifurcations, the following rules suggested by Horton (10, p. 281) were used

1. Starting below the junction, the main stream was projected upstream from the bifurcation in the same direction. The stream joining the main stream at the greatest angle was taken as the lower order (Fig. 1).

2. If both streams were at about the same angle to the main stream at the junction, the shorter was taken as the lower order.

[c] Slope of the main stream, S_c - slope in % of a line drawn along the longitudinal section of the main channel in such a manner so as to have the same area enclosed by it as does the profile (Fig. 1).

Development of an Empirical Graph.—The hydrographs and stage graphs collected from each watershed were reduced to distribution graphs (unit hydrographs modified to show the proportional relation of their ordinates expressed as percentages of the total surface-runoff volume) to eliminate the natural differences in discharge rates arising from watershed size. Where

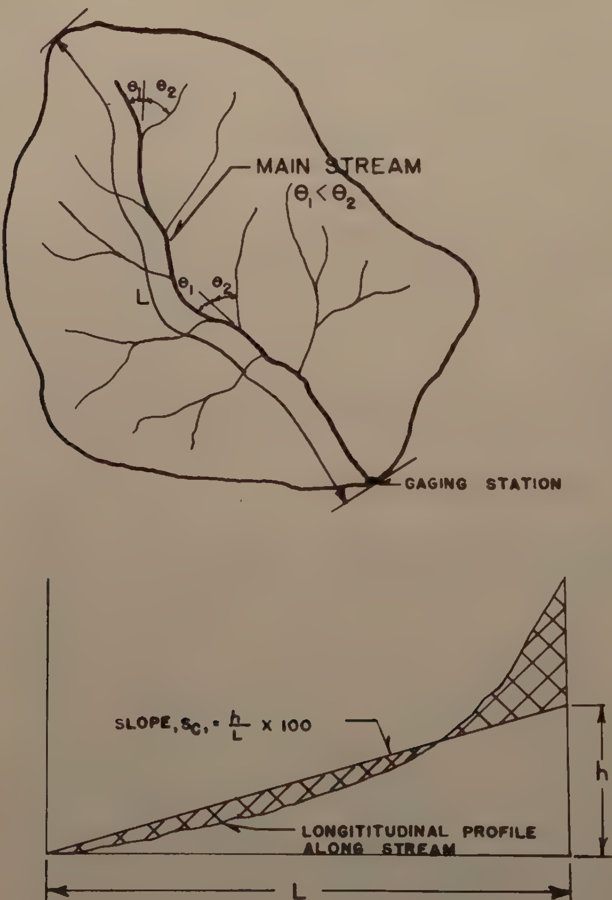

FIG. 1.—WATERSHED CHARACTERISTICS

several distribution graphs are available for a given watershed, a representative graph may be resolved by using recommended procedures (12, 13, and 22).

In this study, the method used was influenced by the inconsistency of the original data. The times-of-occurrence and magnitudes of the peak discharge rates were considered the most significant factors. When the individual graphs, plotted with a common time of beginning of surface runoff, showed small time variations at the peak discharge, an average graph was obtained by finding the

average peak stage and time and sketching a mean graph to conform to the individual graphs as closely as possible (12). If, on the other hand, the composite plot indicated extreme horizontal scattering so as to restrict the graphic determination of an average peak, the graphs were positioned to a location of best fit by giving preference to the following properties in decreasing order of importance; maximum ordinate, time of occurrence of precipitation excess,

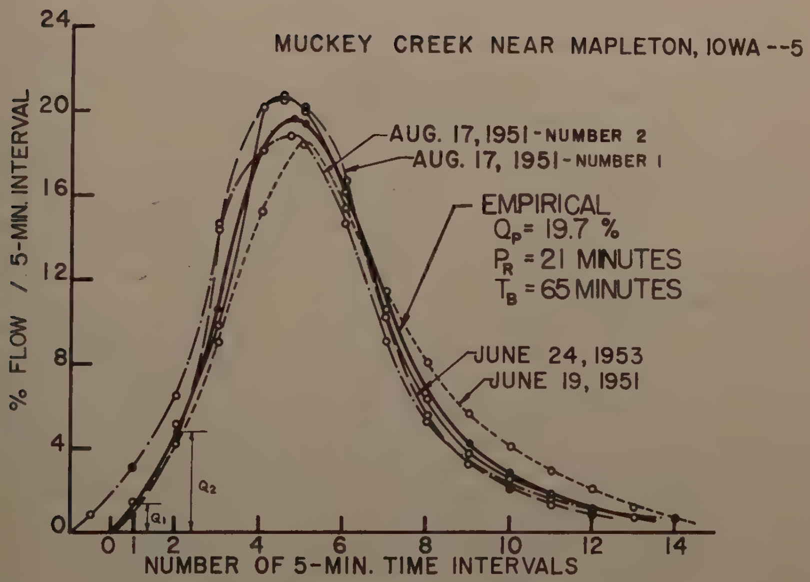

FIG. 2.—EMPIRICAL GRAPH FOR WATERSHED 5 DEVELOPED FROM 4 SELECTED STORMS

ascending limb of the hydrograph, and descending limb of the hydrograph. The average period of rise and peak discharge were then obtained and a representative graph constructed by successive trial plottings.

The representative distribution graph of a watershed was designated the empirical graph. The term "empirical" was adopted to infer that the graph was developed from empirical data and to avoid the possibility of misinterpretation conveyed by the words mean or average.

The empirical graph for watershed 5 developed from 4 individual storms is shown in Fig. 2. It should be noted in Fig. 2 that the time scale has been transposed to coincide with zero discharge and zero time of the empirical graph. The discharge readings are plotted at the midpoints of the time interval that each represents. That is, Q_1, the percentage of total flow occurring during the first 5-min time increment, 0-5 min, is plotted at 2.5 min. Similarly, Q_2, the percentage of total flow occurring during the second 5-min time increment, 5 min to 10 min, is plotted at 7.5 min, and so on. The peak discharge was always computed irrespective of whether or not it fell at the midpoint of an interval. For this watershed, the agreement of the individual graphs is very good, and little difficulty arose in developing the empirical graph. For certain other watersheds, however, the separate graphs exhibited considerable scatter. For those cases, the development of the empirical graph was somewhat subjective, relative to locating the position of best fit.

Development of Dimensionless Graphs.—The empirical graphs for the 42 watersheds were further modified to a standardized form to avoid inconsistencies in the time-increments used in their description. Each graph was adjusted with its ordinate values expressed in percentage flow based on a time increment equal to 1/4 the period of rise (% flow/0.25 P_R) and the abscissa as the ratio of any time, t, divided by the period of rise, P_R, the time from beginning of surface runoff to the occurrence of peak discharge (Fig. 3). The empirical graphs described in this manner were referred to as dimensionless graphs.

The time-increment duration of 0.25 P_R was chosen for the following reasons: (1) The period of rise was ascertained to be an important time characteristic of a given watershed, (2) The use of 0.25 P_R enables definition of the rising limb at four points and (3) The shape of the hydrograph was retained by using this sized increment.

Fitting of the Dimensionless Graph to the Two-Parameter Gamma Distribution.—The methods presently used to evaluate the parameters of Eqs. 1b and 2 from hydrologic data are somewhat limiting. Nash (15) suggests that the method of moments can be used to estimate the parameters, k and n, provided both rainfall and runoff records are available. This procedure is, however, a cumbersome and laborious task. Edson (6) has given a nomograph for determining the parameters of Eq. 1a from the time of occurrence and magnitude of the peak discharge rate of a known unit graph. The obvious limitation in using the nomograph is that it does not utilize all the experimental data in the estimation but considers only the peak ordinate.

Efficient estimates of the unit-graph parameters (Eqs. 1b and 2) can be obtained if consideration is given to the analogy between the unit graph and the two-parameter or incomplete gamma distribution. The equation of the skew statistical frequency curve is given by the relationship

$$f(x) = \frac{N(\gamma)^q}{\Gamma(q)} e^{-\gamma x} x^{q-1} \quad \ldots\ldots\ldots\ldots (3)$$

in which f(x) is any "ordinate" value; x is any "x" value; N represents total frequency or number of observations of x; q and γ are shape and scale parameters, respectively; Γ denotes the gamma function; and e is the base of natural logarithms. Obviously, Eqs. 1b, 2, and 3 define the same curve when the following equalities exist, $f(x) = Q_t$; $N = V$; $x = t$; $q = m = n$; and $\gamma = y = 1/k$. Because of the similarity, it follows that the statistical procedures applied i

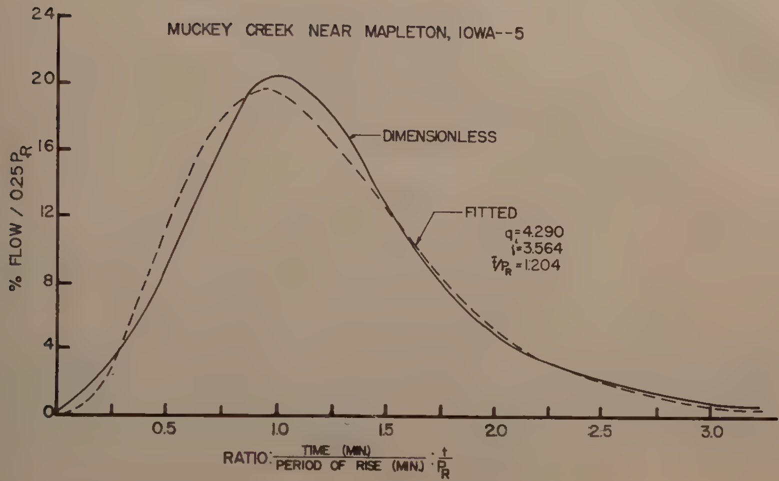

FIG. 3.—DIMENSIONLESS GRAPH AND FITTED TWO-PARAMETER GAMMA DISTRIBUTION FOR WATERSHED 5

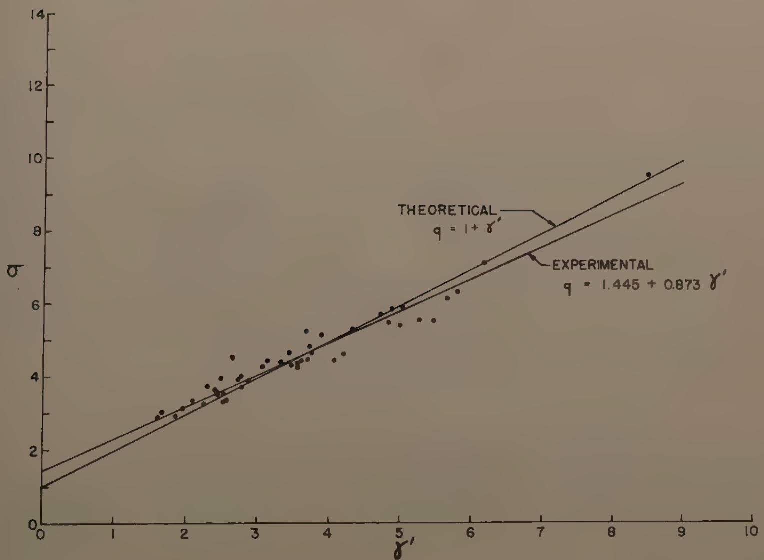

FIG. 4.—THEORETICAL AND EXPERIMENTAL RELATIONSHIPS BETWEEN PARAMETERS q AND γ' OF DIMENSIONLESS GRAPHS

fitting the two-parameter gamma distribution to experimental data can be used to obtain estimators of q and γ for the unit hydrograph.

Because the dimensionless graph is simply a modified form of the unit graph, Eq. 3 can be used to describe its geometry after appropriate changes have been made to the constants. Unlike the unit graph, the value of N (Eq. 3) for the dimensionless graph remains constant, independent of watershed size. By simple computation, it can be shown that the geometry of the dimensionless graph can be approximated by the relationship (7)

$$Q_{t/P_R} = \frac{25.0\,(\gamma')^q}{\Gamma\,(q)} \left(e^{-\gamma' t/P_R}\right) \left(\frac{t}{P_R}\right)^{q-1} \quad \ldots \quad (4)$$

in which Q_{t/P_R} = % flow/0.25 P_R at any given t/P_R value; γ' = dimensionless parameter equal to the product, $\gamma\ P_R$; and q, Γ and e = as described under Eq. 3. According to Eq. 4, with the values of q, γ', and P_R known, the dimensionless graph, distribution graph, and unit graph for the watershed can be developed.

The dimensionless graph of each watershed was processed through an IBM 650 computer to obtain the maximum likelihood estimators of the parameters, q and γ' (8). The experimental and fitted curves for watershed 5 are shown in Fig. 3.

A statistical, Chi-square test was applied to each fitted curve in an attempt to evaluate the goodness of fit. In no case was a significant Chi-square value ($P \lesssim 0.05$) obtained to reject the hypothesis that the fitted and actual curves are of the same population. However, in some cases, their agreement was sufficiently poor, particularly within the crest segment, to invalidate the use of the fitted curve from a hydrologic aspect. (A complete set of figures showing the fitted and actual curves for each watershed, complete with detailed discussions concerning problems involved in the fitting process and in evaluating the goodness of fit in terms of hydrologic acceptance, are given elsewhere (7).) As a result, an arbitrary "point" criterion was established; whereby, the values of q and γ' from the fitted curves that agreed within 20% of the dimensionless graph at the peak ordinate were used. This procedure reduced the number of watersheds in later investigations to 33.

Relation Between the Parameters, q and γ'.—The parameters, q and γ', of Eq. 4 describing the dimensionless graph are linearly related. This relationship can be developed considering that at the peak, $\frac{dQ_{t/P_R}}{d\,(t/P_R)} = 0$, $t/P_R = 1$, and Q_{t/P_R} is a maximum. By setting the first differential of Eq. 4 equal to zero and substituting $t/P_R = 1$ into the result, it follows that

$$q = 1 + \gamma' \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

Eq. 5 states that for the dimensionless graph the variables plot as a straight line with an intercept value and slope equal to unity.

As shown in Fig. 4, the experimental results deviate somewhat from the theoretical expression. The least squares line fitted to these data is defined by the regression

$$q = 1.445 + 0.873\ \gamma' \quad \ldots\ldots\ldots\ldots \quad (6)$$

The difference between the theoretical and experimental curves is indicative of the inability of the fitted graphs to achieve proper positioning of the peak ordinate. Although this difference was found to be statistically significant, it was not considered of sufficient magnitude to restrict the validity of the fitted curves, particularly if consideration is given to the subjectiveness involved in positioning the initial graphs and the scatter of the original data. In practical applications, Eq. 5 would always be used to obtain the peak at a value of $t/P_R = 1$.

Evaluation of the Storage Factor, P_R/γ', from the Watershed Characteristic, $L/\sqrt{S_C}$.—Use of the method of maximum likelihood as a fitting procedure provides that the variables, q, γ', and $\bar{t}/P_R$, are related in the following manner

$$\frac{q}{\gamma'} = \frac{\bar{t}}{P_R} \qquad (7a)$$

in which $\bar{t}/P_R$ is the mean value of the ratio for the dimensionless graph (19). By substituting $\gamma = \gamma'/P_R$ into Eq. 7a, it follows that

$$\frac{q}{\gamma} = E \qquad (7b)$$

in which γ is the scale parameter of the unit hydrograph (Eq. 3) and has the dimensions of the reciprocal of time.

In order to expedite the desired correlation, it is helpful to consider the relationship between the storage constant, k, (Eq. 2) and the variables P_R and γ' for the instantaneous unit graph. From the equalities previously listed, it can be shown

$$k = \frac{1}{\gamma} = \frac{1}{\dfrac{\gamma'}{P_R}} = \frac{P_R}{\gamma'} \qquad (8)$$

The parameter, P_R/γ', like the storage constant, k, measures the storage properties of a watershed or the travel time required for water to pass through a given reach. It was, therefore, referred to as a storage factor.

The magnitude of the storage factor for a given watershed is relatively independent of the effects of rainfall duration. As a consequence, the differences in magnitudes of these factors for different watersheds can be attributed to differences in physical characteristics. R. K. Linsley, F. ASCE, (11) and C. O. Clark, M. ASCE, (4) have shown the storage constant, k, to be related to the watershed characteristics, area, length and slope of the main stream. In addition, Linsley points out that the relationship may be influenced by regional differences.

From purely hydraulic considerations, the magnitude of the storage factor, P_R/γ', would be expected to vary directly with the length of the stream and inversely with some power of the channel slope. Thus, an attempt was made to relate P_R/γ' with the watershed parameter, $L/\sqrt{S_C}$, using data taken from 33 selected watersheds.

With the variables plotted on rectangular coordinate paper, it is evident that their relation is curvilinear. This property may possibly be attributed to use

of the square root of channel slope rather than some other power. J.C.I. Dooge, M. ASCE, (5) suggests that in loose boundary hydraulics, under given conditions of channel flow, travel time varies inversely with the cube root of channel slope. In view of these considerations, the functional form of the relation between the variables was assumed to be that of a power equation

$$\frac{P_R}{\gamma'} = a \left(\frac{L}{\sqrt{S_C}}\right)^b \quad \text{. . . . . (9)}$$

in which a and b, the coefficient and exponent, are evaluated from the experimental data.

Figs. 5, 6, and 7 show the storage factors, P_R/γ', and watershed parameters, $L/\ S_C$, for the 33 watersheds plotted on logarithmic paper according to three regional groupings, Nebraska-Western Iowa, Central Iowa-Missouri-Illinois and Wisconsin, and Ohio. The regression equations for these data computed by the method of least squares are

Nebraska-Western Iowa.—

$$\frac{P_R}{\gamma'} = 7.40 \left(\frac{L}{\sqrt{S_C}}\right)^{0.498} \quad \text{. . . . . (10)}$$

Central Iowa-Missouri-Illinois-Wisconsin.—

$$\frac{P_R}{\gamma'} = 9.27 \left(\frac{L}{\sqrt{S_C}}\right)^{0.562} \quad \text{. . . . . (11)}$$

Ohio.—

$$\frac{P_R}{\gamma'} = 11.40 \left(\frac{L}{\sqrt{S_C}}\right)^{0.531} \quad \text{. . . . . (12)}$$

in which P_R/γ' is in minutes, L in miles and S_C in per cent. For each group, the regression was found to be highly significant.

Usually the standard deviation from regression is used as a measure of the average deviation of the individual points from the regression line (17). However, when using logarithms, it is more convenient to use the coefficient of variation, CV, the ratio of standard deviation from regression to the mean "y"-value, expressed in per cent, as a measure of this deviation (21). The values of CV for the three regression lines were computed as 28.0%, 30.7% and 29.1% respectively. An additional index of the degree of association between the variables is given by the correlation coefficient, r, listed on Figs. 5, 6, and 7.

An analysis of covariance of these data substantiated the selected grouping. This analysis showed the regression of P_R/γ' values on $L/\sqrt{S_C}$ values for the Nebraska-Western Iowa watersheds to be significantly different from that for the Ohio watersheds. The experimental data follow parallel lines that pass through the mean logarithmic values of each group.

In contrast, the data from watersheds within Central Iowa, Illinois, Missouri and Wisconsin adopt positions that correspond to each of the preceding tw

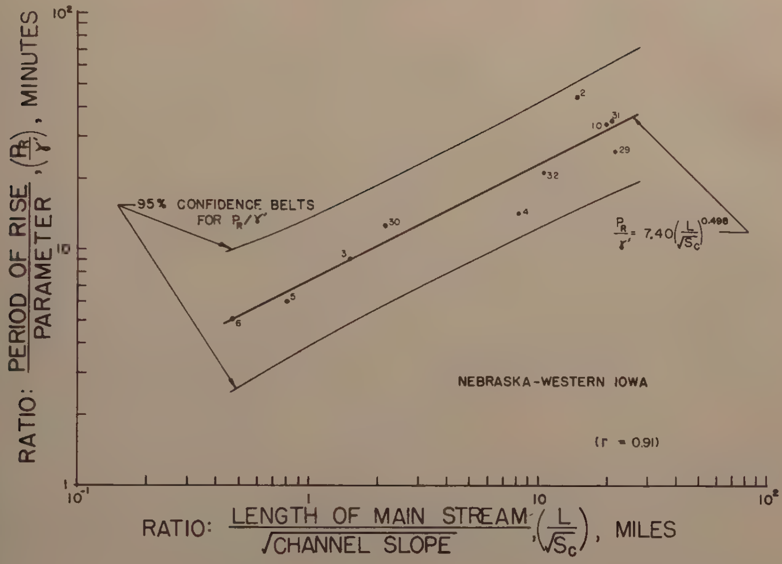

FIG. 5.—RELATION OF STORAGE FACTOR, P_R/γ', AND WATERSHED PARAMETER, $L/\sqrt{S_c}$, FOR WATERSHEDS IN NEBRASKA-WESTERN IOWA

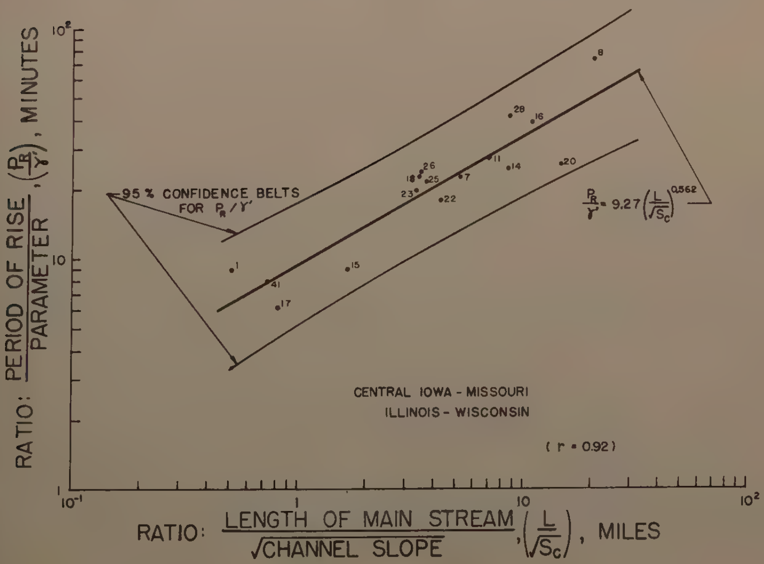

FIG. 6.—RELATION OF STORAGE FACTOR, P_R/γ', AND WATERSHED PARAMETER, $L/\sqrt{S_c}$, FOR WATERSHEDS IN CENTRAL IOWA-MISSOURI-ILLINOIS-WISCONSIN

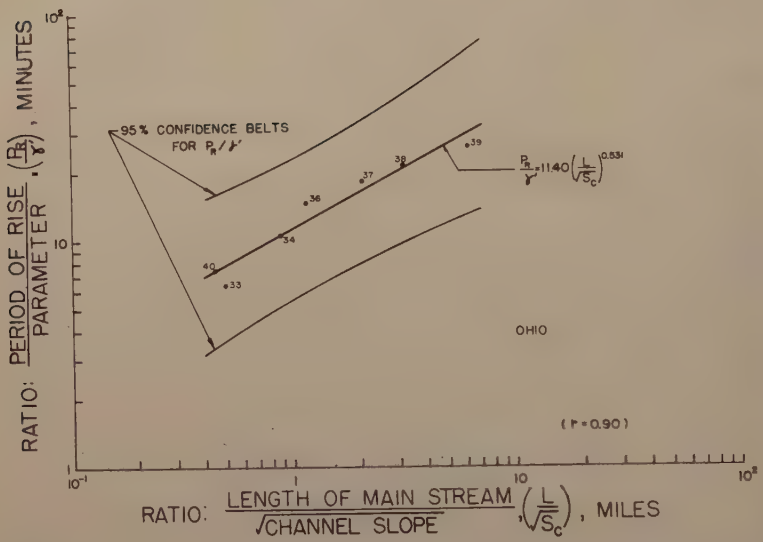

FIG. 7.—RELATION OF STORAGE FACTOR, P_R/γ', AND WATERSHED PARAMETER, $L/\sqrt{S_c}$, FOR WATERSHEDS IN OHIO

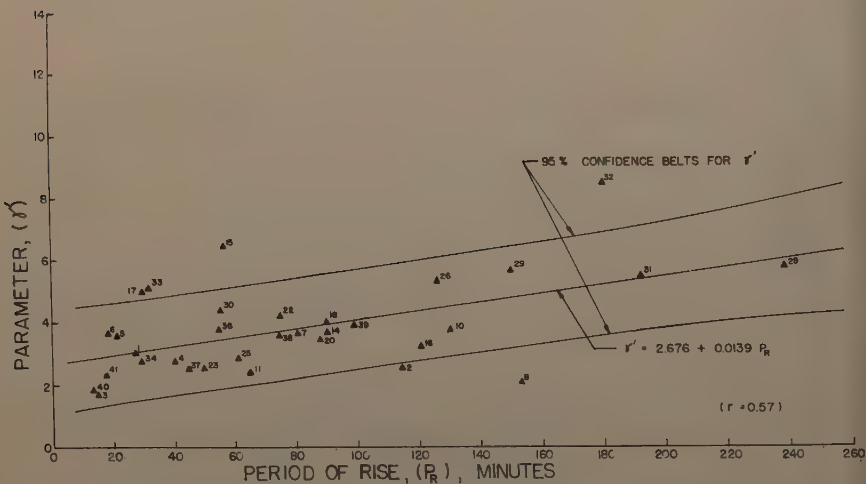

FIG. 8.—RELATION OF PARAMETER, γ', AND PERIOD OF RISE, P_R, FOR 33 SELECTED WATERSHEDS

groups. From a statistical aspect, these could not be separated. However, because of the inability to associate the physical characteristics of the watersheds within these areas with either of the other regions, they were retained separately as an individual group shown in Fig. 6.

Comments.—From Figs. 5 and 7, it is evident that at a common value of $L/\sqrt{SC}$, the storage factor, P_R/γ', is higher for the Ohio watersheds than those for Nebraska-Western Iowa. This difference can be associated with differences in the geometry of the stream channels in the two regions. In Ohio, low flows are confined to shallow, vee-shaped channels that top to narrow, rounded valley bottoms. Even in the case of small flood waves, characteristic of those originating from a unit storm, overbank storage would probably be appreciable. In contrast, stream channels in the loessial area are in the form of deeply-entrenched, U-shaped gullies. For these areas, most flood flows would be confined within the channel. Also, within the range of watersheds studied, a given value of $L/\sqrt{SC}$ is representative of a larger watershed in Ohio than in Nebraska-Western Iowa (7).

In applying the results, it is recommended that the empirical relation be selected from the group with comparable geologic, physiographic, and climatic conditions as the watershed in question. The 95-% confidence belts have been added to Figs. 5, 6, and 7 to facilitate the use of Eqs. 10, 11, and 12 as prediction equations.

Relation of Period of Rise, P_R, and Parameter, γ'.—The results presented give a relationship between dimensionless-graph properties and a relationship between these properties and basin characteristics. They may be expressed in functional form as

$$q = \phi(\gamma') \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

and

$$\frac{P_R}{\gamma'} = \phi'\left(\frac{L}{\sqrt{SC}}\right) \quad \ldots\ldots\ldots\ldots \quad (14)$$

in which ϕ and ϕ' designate the function. With the value of $L/\sqrt{SC}$ known, Eqs. 13 and 14 contain three unknowns. Thus, an additional expression is required before a solution can be obtained.

It was found that the variation in γ' could be significantly explained by linear regression with P_R (Fig. 8). The equation of the line fitted to these data by the method of least squares is

$$\gamma' = 2.676 + 0.0139\ P_R \quad \ldots\ldots\ldots\ldots \quad (15)$$

For the regression the standard deviation from regression was computed as 1.253.

Because Eq. 15 is to be used in conjunction with Eqs. 10, 11, and 12, it is more convenient for computational purposes to express the result in the form

$$\frac{P_R}{\gamma'} = \frac{1}{\frac{2.676}{P_R} + 0.0139} \quad \ldots\ldots\ldots\ldots \quad (16)$$

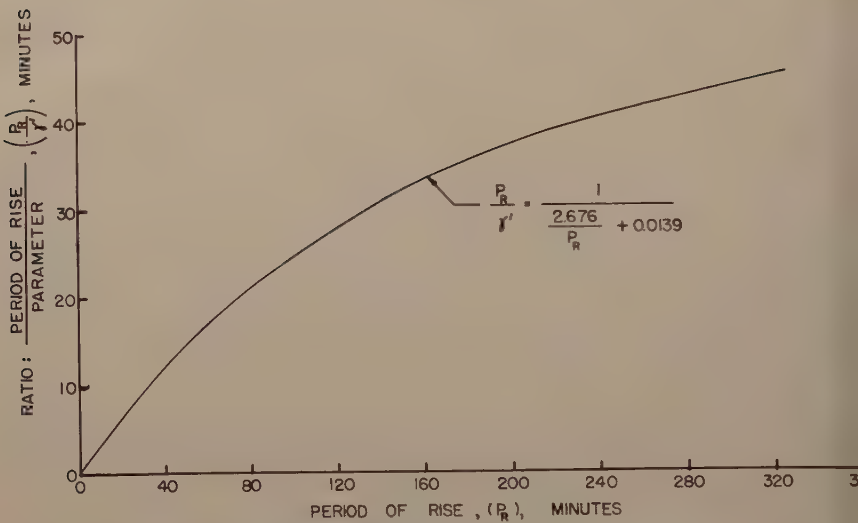

FIG. 9.—RELATION OF STORAGE FACTOR, P_R/γ', AND PERIOD OF RISE, P_R

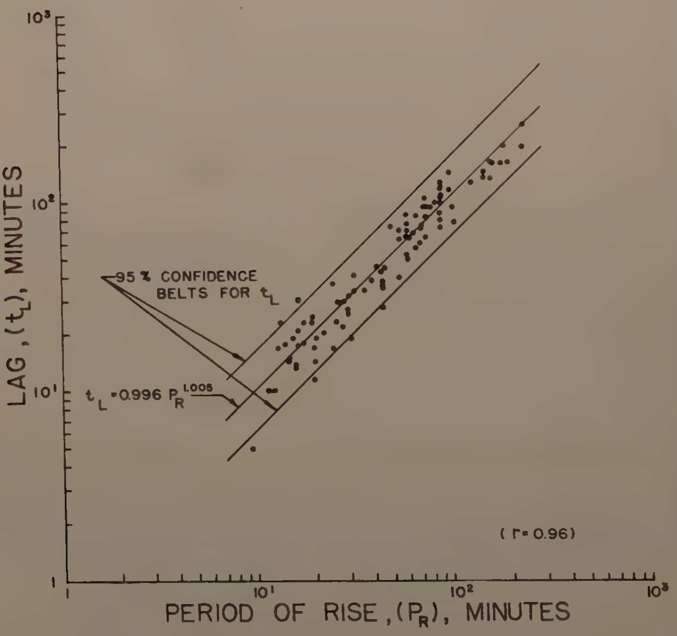

FIG. 10.—RELATION OF LAG, t_L, AND PERIOD OF RISE, P_R, FOR 94 SELECTED STORMS

Eq. 1b is shown plotted in Fig. 9 that can be used to solve for P_R with P_R/γ' known.

Comments.—The use of the results reported in Figs. 5, 6, 7, and 9 should be limited to watersheds having characteristics that fall within the limits of the experimental data. Additional study indicated that best reproducibility was obtained on the watersheds with $L/\sqrt{SC}$ values less than 7 miles. It is evident from Eq. 1b that as P_R approaches infinity, P_R/γ' approaches a maximum value of 71.9 min. By using additional results given by Gray (7), it can be shown that this limit prohibits the use of the regressions to watersheds of the approximate sizes given in Table 2.

Selection of a Time Parameter.—Before applying the synthetic unit graph to a given storm sequence, it is necessary to have available a time parameter relating the salient features of rainfall and runoff for the area in question. Several forms of lag have been proposed for this purpose.

One of the more common forms of lag was introduced by F. F. Snyder, F. ASCE, (18) in 1938. He defined lag as the time difference between the center of mass of a surface-runoff-producing rain and the occurrence of peak discharge. For lag defined in this manner, in order to obtain a constant lag value for a given watershed, it is necessary to specify the storm type; otherwise,

TABLE 2.—APPROXIMATE MAXIMUM WATERSHED SIZE FOR WHICH THE PREDICTION EQUATIONS ARE APPLICABLE

Region (1)	Watershed area, in square miles (2)
Nebraska-Western Iowa	362
Central Iowa-Missouri-Illinois-Wisconsin	94
Ohio	82

due to the unsymmetrical nature of the hydrograph, the magnitude of lag of a given basin will vary with storm duration.

In this study lag, as defined by Snyder, was adopted. No restriction was placed on the storm type other than the unit-storm criteria, however. An initial attempt to determine the lag of each watershed proved unsuccessful because of the lack and inadequacy of precipitation data. As a result, an additional study was undertaken to find a more suitable time parameter, one which could be obtained for each watershed.

As shown by the Soil Conservation Service (20), lag, t_L, and time of concentration of the watershed, T_c, are related in the form $t_L = 0.60\ T_c$. For watersheds of the size used in the study, it is reasonable to assume P_R approximates T_c. It follows from there, that t_L and P_R would be related. These variables from 94 selected storms are shown plotted on logarithmic paper in Fig. 10. The regression line fitted to these data by the method of least squares is defined by the equation

$$t_L = 0.996\ P_R^{1.005} \quad \ldots\ldots\ldots\ldots\ldots \quad (17)$$

For all practical cases, the values of the constant and exponent of Eq. 17 can be taken as unity, in which case $t_L = P_R$. That is, a given change in P_R pro-

duces an equal change in t_L. This simple linear regression between t_L and P_R found in the data conforms to the form of the relationship suggested in previous comments.

A similar result was obtained by R. B. Hickok, F. ASCE, R. V. Keppel, and B. R. Rafferty (9) in their studies of rainfall and runoff records from 14 experimental watersheds in Arizona, New Mexico, and Colorado. They reported (9, p. 615)

> "Rise time varied from 74 per cent to 145 per cent of the lag time (time from the center of mass of a limited block of intense rainfall to the resulting peak of the hydrograph) for the individual watersheds in this study. The average for all watersheds was 102 per cent."

TABLE 3.—COORDINATES OF THE SYNTHESIZED UNIT HYDROGRAPH

$\frac{t}{P_R}$ (1)	Accumulated time, in min (2)	% flow[a] 0.25P_R (3)	Cumulative % flow, 0.25P_R (4)	Unit graphs, in cfs (5)
0.000	0.0	0.0	0.0	0
0.125	7.3	0.3	0.3	40
0.375	21.8	5.2	5.5	694
0.625	36.3	13.0	18.5	1,736
0.875	50.8	17.6	36.1	2,350
1.000	58.0	---	---	2,430[b]
1.125	65.3	17.7	53.8	2,363
1.375	79.8	14.9	68.7	1,989
1.625	94.3	11.2	79.9	1,495
1.875	108.0	7.7	87.6	1,028
2.125	123.3	5.0	92.6	668
2.375	137.8	3.1	95.7	414
2.625	152.3	1.9	97.6	254
2.875	166.8	1.1	98.7	147
3.125	181.3	0.6	99.3	80
3.375	195.8	0.3	99.6	40
3.625	210.3	0.2	99.8	27
3.875	224.8	0.1	99.9	14
4.125	239.3	0.1	100.0	13
		Total	100.0	13,352

[a] Rounded to nearest 0.10%.
[b] Peak discharge rate; not included in total.

The association between the lag time used in the preceding and lag as used herein is assumed. For short-duration storms, as used in the development of unit graphs for small watersheds, the center of a limited block of intense rain and the mass center of the surface-runoff-producing rain would be nearly coincident. The variances from regression of the t_L values and P_R values were approximately equal. The coefficients of variation were computed to be 27.1 and 25.7%, respectively.

On the basis of this evidence, it was concluded that the period of rise, P_R of the hydrograph could be used as an effective time parameter to relate th

salient features of rainfall and runoff on a given watershed. The result is generally applicable only for uniformly-distributed, short-duration, high-intensity storms occurring over small watershed areas.

Application of Results.—

Problem.—Define the unit hydrograph for a watershed, 5 sq miles in area, that falls within a region of comparable geologic, physiographic, and climatic

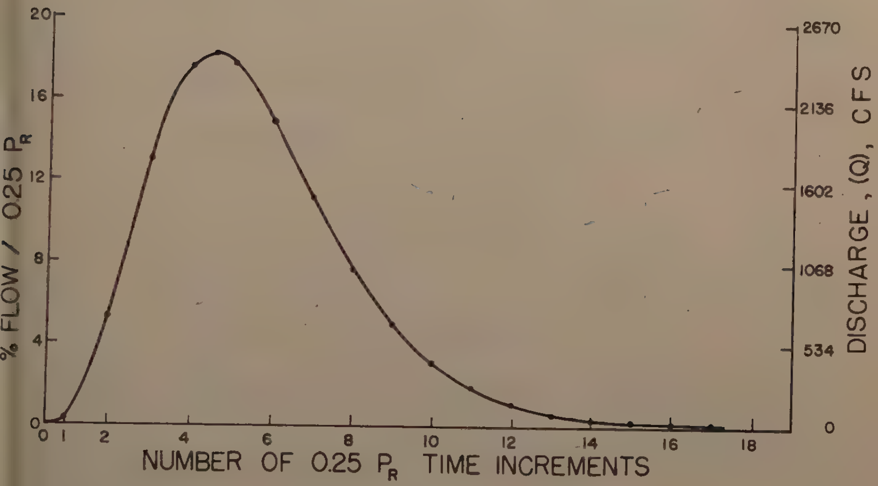

FIG. 11.—SYNTHETIC UNIT GRAPH FOR 5-SQ-MILE WATERSHED USED IN ILLUSTRATIVE PROBLEM

conditions as those of Western Iowa. The following information was obtained from an available topographic map: L = 3.80 miles and S_C = 0.57%.

Procedures.—

Step 1. Determine parameters; P_R, γ' and q.

A. With $L/\sqrt{S_C} = 3.80/\sqrt{0.57} = 5.03$ miles, enter Fig. 5 and select; $P_R/\gamma' = 16.6$ min.

B. With $P_R/\gamma' = 16.6$ min, enter Fig. 9 and obtain; $P_R = 58$ min. Therefore, $\gamma' = 58/16.6 = 3.494$

C. Set the peak to fall at $t/P_R = 1$, by substituting $\gamma' = 3.494$ into Eq. 5 and solve for q = 4.494.

Step 2. Compute the ordinates of the dimensionless graph.

A. Using Eq. 4, compute the % flow/0.25 P_R at the respective values of t/P_R - 0.125, 0.375, 0.625 . . . and every succeeding increment of t/P_R = 0.250, until the sum of the ordinates approximates 100% (Table 3). Also

compute the peak percentage. At the peak,

$$Q_{(1)} = \frac{25.0\ (3.494)^{4.494}}{\Gamma\ (4.494)}\ e^{-3.494(1)}\ (1)^{4.494} = 18.2\%$$

Step 3. Develop the unit hydrograph.
A. Compute the necessary conversion factor.
(a) Volume of unit hydrograph, V

$V = 1$ in. x 5 mile^2 x 640 acre per mile^2 x $\frac{1}{12 \text{ in. per ft}}$ x 43560 ft^2 per acre = 11,616,000 ft^3

(b) Volume of dimensionless graph, V_D

$V_D = \Sigma$ cfs x 0.25 x 58 min x 60 sec per min = 870 Σ cfs - sec. Because the two volumes, V and V_D, must be equal, it follows that Σ cfs = 11,616,000/870 = 13,352 cfs.

B. Convert the dimensionless graph ordinates to cfs.

$$\frac{Q_t}{P_R} = \left(\frac{\%\ \text{flow}/0.25\ P_R}{100}\right) \Sigma\ \text{cfs}$$

Therefore, at the peak,

$$Q_p = 18.2/100 \times 13{,}352 = 2{,}430 \text{ cfs.}$$

C. Convert the time base of the dimensionless graph to absolute time units. At the peak, $t/P_R = 1$; therefore, $t = 58$ min.

Step 4. Plot the unit hydrograph (Fig. 11).

According to Fig. 10, the time of beginning of surface, runoff should be placed coincident with the centroid of precipitation. For convenience of computation, the unit hydrograph should be associated with unit-storm periods of 0.25 P_R duration.

CONCLUSIONS

A workable synthetic procedure for the development of the unit hydrograph for small watersheds from measurable physical characteristics is presented.

Hydrologic and topographic data from 42 small watersheds in the states of Illinois, Iowa, Missouri, Ohio, and Wisconsin were analyzed. For each watershed, a representative distribution graph, the so-called empirical graph, was derived and modified to a dimensionless form based on the period of rise, P_R, as the time parameter.

The two-parameter gamma distribution described by the parameters, q and γ', was fitted to each dimensionless graph and the maximum likelihood estimators of the parameters obtained. Relationships were established so that the parameters, P_R, q and γ' could be evaluated from the topographic characteristics L and S_C of a given basin. With P_R, q and γ' known, the dimensionless graph, distribution graph, and unit hydrograph for the basin can be described.

The following conclusions were derived from this study.

1. In general, the two-parameter gamma distribution can be used to describe the dimensionless graph, distribution graph, or unit hydrograph.

2. The storage factor, P_R/γ', can be predicted with reasonable success from the watershed factor, $L/\sqrt{S_C}$, provided consideration is given to regional influence.

3. The parameter, γ', of the two-parameter gamma distribution describing the dimensionless graph can be estimated from the period of rise.

4. For a given watershed, the dimensionless graph, distribution graph, and unit hydrograph can be derived from the watershed characteristic, $L/\sqrt{S_C}$;

ACKNOWLEDGMENTS

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MANIFOLD STILLING BASIN

By Gene R. Fiala,[1] M. ASCE and Maurice L. Albertson,[2] F. ASCE

SYNOPSIS

This paper presents the manifold stilling basin as a device for dissipating excess kinetic energy. This device has certain important advantages for some conditions. Two successful field installations of the manifold stilling basin have already been made. Both installations were made at the outlet ends of pipe drops in canals in which the vertical drop of the water surface was several tens of feet.

INTRODUCTION

The dissipation of kinetic energy in hydraulic structures can take place with flow either in the horizontal direction or in the vertical direction, or in a combination of both horizontal and vertical directions. M. L. Albertson and G. I. Smith,[3] M. ASCE, have analyzed this breakdown in some detail as illustrated in Fig. 1. From this figure it is evident that energy is dissipated with flow in the:

1. Horizontal direction by
   a. Shear drag,

Note —Discussion open until December 1, 1961. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Hydraulics Division, Proceedings of the American Society of Civil Engineers, Vol. 87, No. HY 4, July, 1961.

[1] Hydr. Engr., U. S. Bur. of Pub. Roads, Portland, Ore.

[2] Prof. of Civ. Engrg. and Dir. of the Research Foundation, Colorado State Univ., Fort Collins, Colo.

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b. Pressure drag,
c. Increase in piezometric head.

2. Vertical direction by
   a. Diffusion of jets vertically downward, and
   b. Diffusion of jets vertically upward.

In this paper, consideration is given only to the dissipation of kinetic energy (with flow in the vertically-upward direction) by means of a manifold type of structure, that is illustrated schematically in Fig. 1(d). As outlined later in some detail, the use of a jet issuing vertically upward into the overhead tailwater has two important advantages:

1. The jet entrains a part of the surrounding fluid, and in so doing it distributes its energy throughout a greater mass. Furthermore, much of the kinetic energy is converted into heat from the resulting shear, either directly, or indirectly by the creation of relatively fine-grain turbulence.
2. The kinetic energy that remains in the diffused jet as it reaches the surface of the tailwater causes the jet to rise in a boil above the tailwater. The boil then spreads radially, causing a rapid reduction and dispersion of kinetic energy.

This type of device (Fig. 1(d) and Fig. 2,) was originally suggested by the second writer, and a laboratory model study was made to determine its effectiveness. Two field installations were later found to be so effective and economical it was decided to conduct a generalized laboratory investigation to obtain design data. Fiala[4] conducted the investigation and his data are reported in this paper. The design of the manifold stilling basin is based on the manifold principle and the principles of diffusion of submerged jets.

Notation.—The letter symbols adopted for use in this paper are defined where they first appear and are arranged alphabetically, for convenience of reference, in the Appendix.

MANIFOLD APPLICATIONS

Studies of various applications of manifold flow have been conducted by investigators since the early 1900's. A comprehensive study of gas-burner manifolds was conducted by J. D. Keller.[5] The results of his study are of particular importance to this paper because he found that, for rectangular manifolds having a constant width, the contour of the bottom can be assumed to be linear for values of $L/\sqrt{A} \leq 10$, in which L is the length of the manifold and A is the cross-sectional area of the manifold at the inlet end (Fig. 2).

A series of investigations by J. S. McNown[6] and others at the Iowa Institute of Hydraulic Research has shown that certain aspects of the theoretical analysis of manifold flow are not sufficiently accurate for practical use except in a few instances.

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4 "Laboratory study of a manifold stilling basin," by Gene R. Fiala, thesis presented to the Colorado Agricultural and Mechanical College in Fort Collins, Colo., in May 1957, in partial fulfillment of the requirements for the degree of Master of Science.

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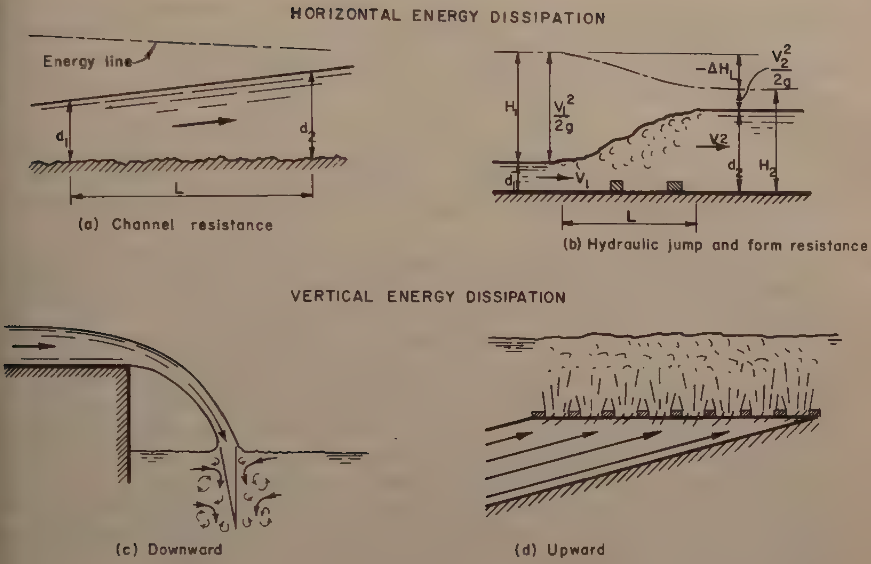

FIG. 1.—METHODS OF DISSIPATION OF KINETIC ENERGY.

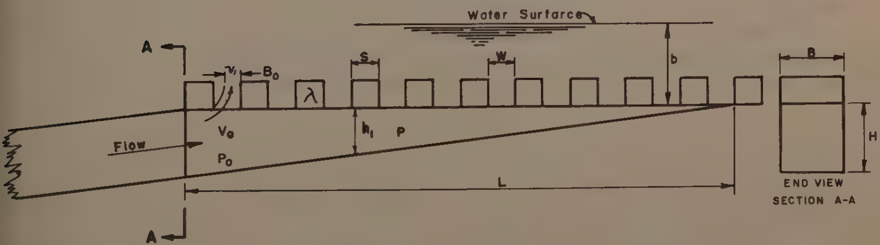

FIG. 2.—DEFINITION SKETCH FOR DIMENSIONAL ANALYSIS

Manifold flow in sprinkling systems for irrigation was summarized by J. E. Christianson[7] in 1942.

Considerable research has been conducted on various aspects of the flow in lock manifolds. As a result of experiments at the Panama Canal Laboratory in 1939 and 1940, E. Soucek, F. ASCE, and E. W. Zelnick, F. ASCE,[8] concluded that the design of a lock manifold is an indirect process so that the design must be assumed, analyzed, and then modified until the desired hydraulic behavior is obtained.

7 "Hydraulics of Sprinkling Systems for Irrigation," by J. E. Christianson, ASCE, Transactions, Vol. 107, 1942, p. 221.

8 "Lock Manifold Experiments," by Edward Soucek, and E. W. Zelnick, ASCE, Transactions, Vol. 110, 1945, p. 1357.

Studies conducted for lock emptying and filling systems by the United States Army Corps of Engineers[9,10] are primarily for manifolds that operate under unusual circumstances. There is only limited application of these studies to manifold problems of a general nature.

BASIC REQUIREMENTS FOR A MANIFOLD DESIGN

The basic requirements for a manifold designed as an energy dissipator are:

1. The velocity and discharge per foot of length from the manifold must be the same at all points along the length of the manifold in order to have a uniform distribution of energy and momentum flux upward.
2. The shape, dimensions, and other design features must be practical, both for economic reasons and in order that the design can be fitted to the problem of construction in the field.

For the generalized laboratory investigation reported herein, consideration was given to various types of manifolds to find one that would best meet the basic requirements. The manifold design selected for this study had a rectangular cross section, a constant width, a linearly-varying height, and an $L/\sqrt{A}$ - ratio of 8. This design is basically the same as one of those discussed by Keller[5] and is similar to the successful prototype stilling basins previously mentioned.

With this manifold design, uniform efflux velocity and discharge can be obtained along the length of the manifold. The linear shape results in economical construction and the dimensions are practical for field installations.

Once the basic features of the design are decided, the variables influencing operation of the design can be grouped and treated by theoretical analysis and dimensional analysis to aid in the laboratory investigation.

THEORETICAL SHAPE OF MANIFOLD

The theoretical analysis for the simple manifold with uniform discharge conditions uses the energy equation

$$\frac{V^2}{2g} + \frac{p}{\gamma} + z = \text{constant} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

and the continuity equation

$$Q = A\,V \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

in which Q is the discharge across the section in the manifold for which A is the cross-sectional area; V denotes the mean velocity across the section; p is the pressure; z represents the evevation; g is the acceleration of gravity

9 Final Report: "Laboratory Tests on Hydraulic Models of Filling and Emptying Systems for New Cumberland Locks, Ohio River, St. Paul, Minn.," U. S. Corps of Engrs. 1952. Iowa Univ. Hydr. Lab., Hydr. Lab. Report No. 56.

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γ denotes the specific weight of the water, and v is the velocity at a point in the manifold.

The conditions that are imposed for manifold operation with a constant velocity and uniform discharge issuing along the manifold are:

1. Constant internal pressure to cause the constant efflux velocity along the length of the manifold.

2. A discharge of Q_o, entering the manifold at the upstream end, which is reduced linearly to a discharge of zero at the downstream end of the manifold.

Because the variation of elevation z in Eq. 1 is relatively small compared with the magnitude of $v^2/(2\,g)$ and p/γ, it can be dropped from consideration. Therefore, with p/γ a constant, Eq. 1 reduces to

$$v = \text{constant} \qquad (3)$$

The discharge issuing from the manifold must be uniform along its length. Therefore, the discharge inside the manifold must vary linearly throughout the length of the manifold. Consequently,

$$Q = K\,S \qquad (4)$$

in which S is the distance from the downstream end of the manifold. When $S = L$, $Q = Q_o$. Hence, Eq. 4 becomes

$$\frac{Q}{Q_o} = \frac{S}{L} \qquad (5)$$

in which L is the length of the manifold. By substituting Q from Eq. 5, and h_1 B for A in Eq. 2; and by considering $V = Q_o/A_o = Q_o/(H\,B)$, Eq. 2 becomes

$$Q_o \frac{S}{L} = h_1\,B\,\frac{Q_o}{H\,B} \qquad (6a)$$

or

$$\frac{h_1}{H} = \frac{S}{L} \qquad (6b)$$

in which h_1 is the height of the manifold at any point; B represents the width of the manifold; and H is the height of the manifold at the entrance. Eq. 6b clearly shows that the shape of the manifold is composed of straight lines, because the height of the manifold varies directly with the distance from the downstream end, and that the shape is independent of flow and fluid properties if boundary resistance is assumed to be negligible.

DIMENSIONAL ANALYSIS

Treatment of this problem of flow into and from a manifold by dimensional analysis is made in order to determine the dimensionless parameters that influence the phenomenon. The laboratory study of the problem can then be planned to determine the actual relationship of these parameters.

A definition sketch is shown in Fig. 2. The variables involved in the problem can be classified into three categories:

1. Variables describing the geometry of the manifold and flow system: L is the length of manifold; B represents the width of manifold; H is the height of manifold at entrance; h_1 denotes the height of manifold at a point; w is the width of opening; s is the size of cross bar; b represents the tailwater depth; x is the distance along the jet; and λ denotes the shape of cross bar.

2. Variables describing the flow: v_o is the mean velocity at entrance to manifold; v_1 represents initial velocity of jet issuing from manifold; v is the velocity at a point in the manifold; v_{max} denotes the centerline velocity in jet at point x; p_o is the pressure at entrance to manifold; p represents the pressure at a point in the manifold; a is the boil height; h is the wave height; and B_o represents the effective initial width of jet issuing from manifold.

3. Variables describing the fluid: $\Delta\gamma$ is the difference in specific weight across the air-water interface; μ represents the dynamic viscosity of water; and ρ is the density of water.

Of the foregoing variables, v_1, p, p_o, a, h_1, v, v_{max}, and B_o can be considered as dependent while the others are independent. By selecting v_1 as the only dependent variable for the moment, and omitting the other dependent variables, application of the Pi-theorem gives

$$\frac{v_1}{v_o} = f_1\left(\frac{L}{B}, \frac{H}{B}, \frac{w}{B}, \frac{s}{B}, \lambda, \frac{b}{B}, \frac{v_o}{\sqrt{\frac{B\,\Delta\gamma}{\rho}}}, \frac{v_o B}{\frac{\mu}{\rho}}\right) \quad \text{(8}$$

in which B, v_o, and ρ are used as repeating variables.

For simplicity in equipment and experimentation, the parameters L/B and H/B can be selected to remain constant during the experiments, and therefore they can be deleted. Furthermore, the length ratios can be rearranged as w/s, b/B, and b/s; and b/B_o can be substituted for b/B and b/s can then be changed to B_o/w. Finally, the Froude number F can be substituted for $v_o/\sqrt{b\,\Delta\gamma/\rho}$ and Reynolds number R for $v_oB/(\mu/\rho)$ to give

$$\frac{v_1}{v_o} = f_2\left(\frac{w}{s}, \frac{b}{B_o}, \frac{B_o}{w}, \lambda, F, R\right) \quad \text{(}$$

In a similar manner, the dimensionless parameters representing the remaining dependent variables are found to be $\Delta p/(\rho\, v_o^2)$ and $p_o/(\rho\, v_o^2)$; and the boil height a and wave height h (which are ultimately two of the principal criteria for design) can be related to the velocity head $v_1^2/(2\,g)$ to give $\frac{a}{v_1^2/(2\,g)}$ and $\frac{h}{v_1^2/(2\,g)}$. Each of these dependent parameters is equal to some function of the same independent parameters, as follows.

$$\frac{\Delta p}{\rho v_o^2} = f_3\left(\frac{w}{s}, \frac{b}{B_o}, \frac{B_o}{w}, \lambda, F, R\right) \quad \text{(}$$

$$\frac{p_o}{\rho v_o^2} = f_4\left(\frac{w}{s}, \frac{b}{B_o}, \frac{B_o}{w}, \lambda, F, R\right) \qquad (11)$$

$$\frac{a}{\frac{v_1^2}{2g}} = f_5\left(\frac{w}{s}, \frac{b}{B_o}, \frac{B_o}{w}, \lambda, F, R\right) \qquad (12)$$

and

$$\frac{h}{\frac{v_1^2}{2g}} = f_6\left(\frac{w}{s}, \frac{b}{B_o}, \frac{B_o}{w}, \lambda, F, R\right) \qquad (13)$$

In one phase of the preliminary testing, a study of $\frac{\Delta p}{\rho v_1^2/2}$ (in which $\Delta p = p - p_o$) was made to ascertain the effect of gravity (as represented by the Froude number) and viscosity (Reynolds number). This pressure parameter is likewise a function of the same variables. When F and R were varied, for given conditions of manifold geometry and tailwater depth over the range in which the experiments were conducted, the variation in the relative pressure change $\Delta p/(\rho v_o^2)$ was negligible except at low tailwater depths. Therefore, it was concluded that the effect of gravity (F) and viscosity (R) was of secondary importance.

The relative thickness of the jet B_o/w (which is a form of contraction coefficient) was assumed to be constant for a given geometry and γ so that

$$\frac{a}{\frac{v_1^2}{2g}} = f_7\left(\frac{w}{s}, \frac{b}{B_o}\right) \qquad (14)$$

A similar analysis can be made for $\frac{h}{v_1^2/(2g)}$ to yield

$$\frac{h}{\frac{v_1^2}{2g}} = f_8\left(\frac{w}{s}, \frac{b}{B_o}\right) \qquad (15)$$

Eqs. 14 and 15 were used to study the variation of relative boil height and relative wave height with respect to the geometry of the manifold and the tailwater depth.

APPLICATION OF SUBMERGED-JET RELATIONSHIPS

Submerged jets are an integral part of energy dissipation in a vertical direction. The characteristics of the mean flow pattern of a single jet submerged in an infinite body of its own fluid were determined analytically and

experimentally by Albertson and others[11] in 1950. Their experimental data provide the necessary empirical coefficients for flow from both slots and orifices. Fig. 3 is a plot of their experimental data for the centerline velocity of the jet flowing from a slot. Plots of the equations and data for the volume-flux ratio, the energy-flux ratio, and the momentum-flux ratio are shown in Fig. 4.

Dissipation of kinetic energy by a manifold stilling basin is brought about by diffusion of the jets of water issuing from the manifold into the tailwater above the manifold. Energy must be dissipated by the stilling basin so that a minimum amount of protection will be required on the bank and bed of the channel downstream. There, the remaining energy of the flow (it is neither necessary nor practical to attempt to dissipate all of the kinetic energy) must be described in some physical manner so that eventually it can in turn be related to the resulting erosion.

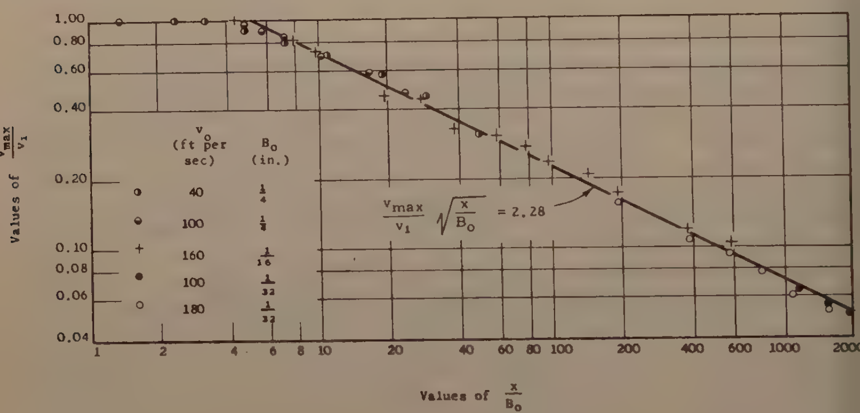

FIG. 3.—DISTRIBUTION OF CENTER LINE VELOCITY FOR FLOW FROM SLOT

Relationship Between Kinetic Energy, Boil Height, Wave Height, and Erosion.— With a manifold stilling basin, erosion of the channel banks (neglecting the be for this study) is primarily a function of wave height h. This is because th initial direction of flow (the jets) is vertically upward and at some distanc from the banks. Through internal shear the velocity is reduced before flo begins outward at the surface of the tailwater. As the flow spreads outwa over a relatively large area, the velocity is further reduced and no high velocity flow is directed against the banks to cause erosion.

The wave height h at the boundary is a function of the boil height a, at whic the boil height is defined as the difference in height between the water surfa over the jet opening and the mean tailwater surface.

Expression For Boil Height.—An expression for boil height a can be deriv in terms of the initial jet velocity v_1, the tailwater depth b, and the effecti

11 "Diffusion of submerged jets," by M. L. Albertson, Y. B. Dai, R. A. Jensen, a Hunter Rouse. Transactions, ASCE, Vol. 115, 1950, p. 639.

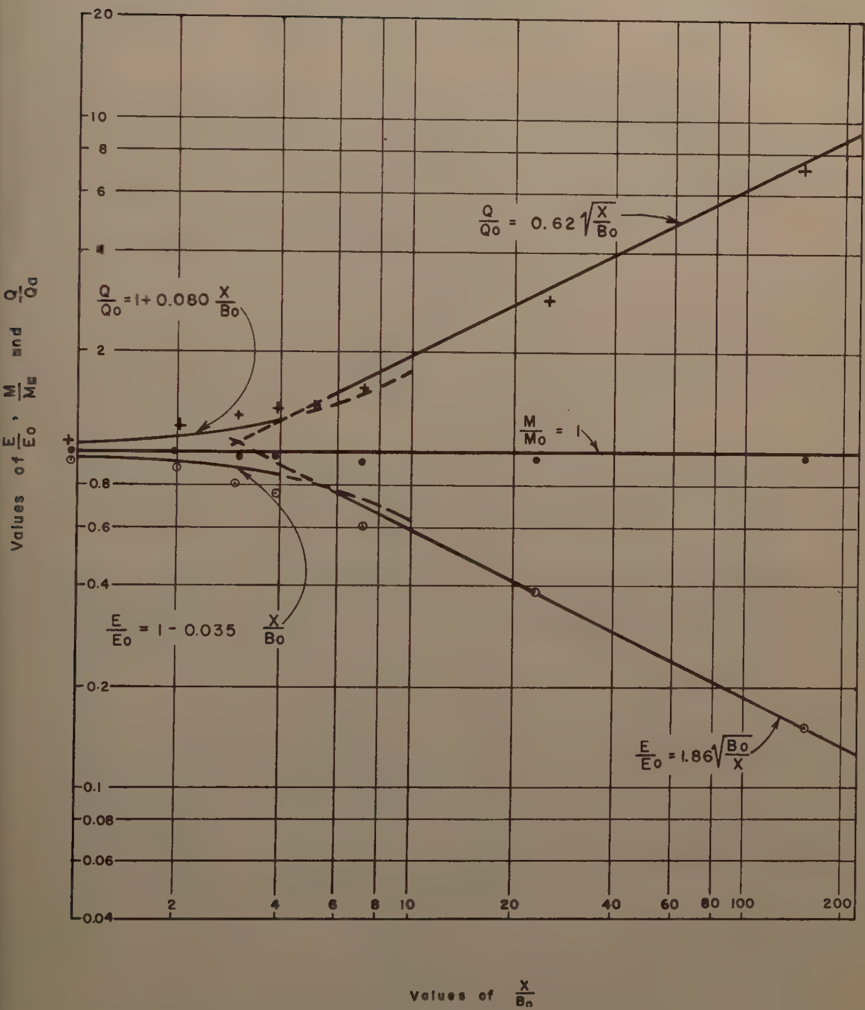

FIG. 4.—DISTRIBUTION OF VOLUME, MOMENTUM, AND ENERGY FLUX DOWNSTREAM FROM SLOT

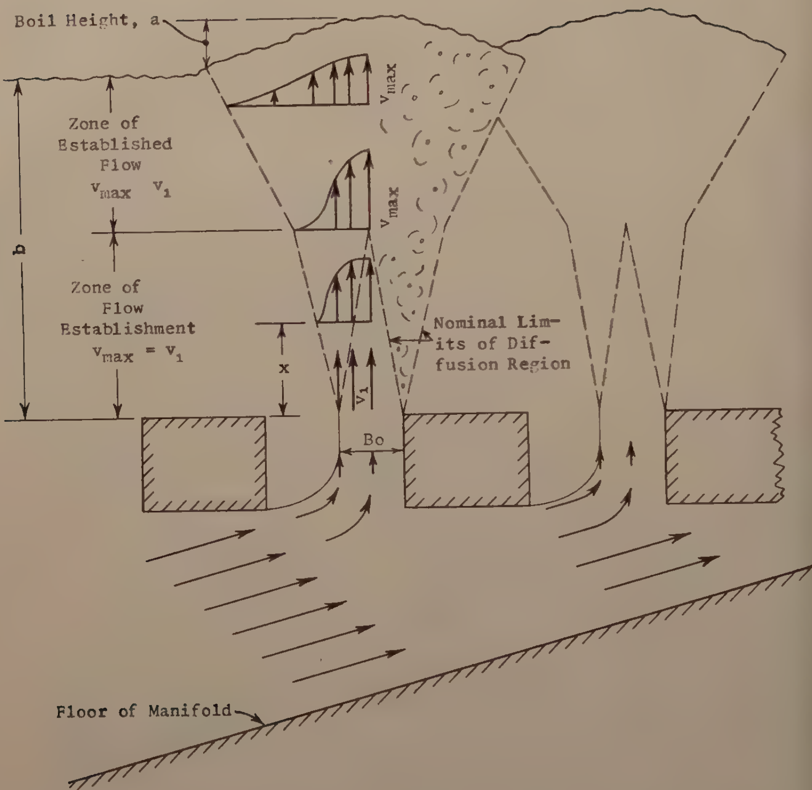

FIG. 5.—DEFINITION SKETCH FOR JET FLOW IN A MANIFOLD STILLING BASIN

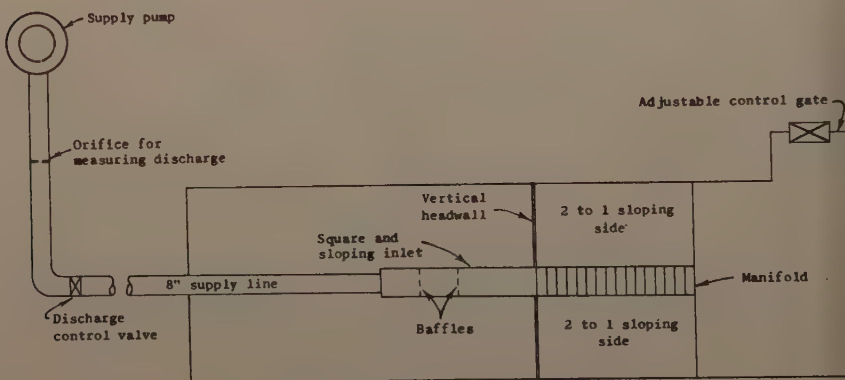

FIG. 6.—PLAN VIEW OF LABORATORY EQUIPMENT

width of jet B_o (Fig. 5). A relationship between wave height h, and boil height a can then be determined experimentally—with the wave height being considered as a qualitative indication of the residual kinetic energy in the manifold stilling basin.

The following assumptions are necessary in deriving the expression for boil height:

1. The jet is diffused according to the theory developed by Albertson and others.[11]
2. The velocity of the jet v_{max} at elevation b above the manifold causes a boil at the water surface that has a height a equal to the remaining velocity head of the jet $(v_{max})^2/(2\,g.)$

The second assumption can be expressed as

$$a = \frac{\left(v_{max}\right)^2}{2\,g} \qquad (16)$$

and the boil height a and the initial velocity head of the jet $v_1^2/(2\,g)$ as a ratio

$$\frac{a}{\dfrac{v_1^2}{g}} = \frac{\dfrac{\left(v_{max}\right)^2}{2\,g}}{\dfrac{v_1^2}{2\,g}} \qquad (17)$$

Eq. 17 can be combined with the equation developed by Albertson and others[11] which expresses the relationship between the velocity of the jet v_{max} (at a distance x from the slot of width B_o) in terms of the initial velocity of the jet v_1.

$$v_{max} = \frac{2.28\,v_1}{\sqrt{\dfrac{x}{B_o}}} \qquad (18)$$

to yield

$$\frac{a}{\dfrac{v_1^2}{2\,g}} = \frac{\dfrac{\left(\dfrac{2.28\,v_1}{\sqrt{\dfrac{x}{B_o}}}\right)^2}{2\,g}}{\dfrac{v_1^2}{2\,g}} \qquad (19)$$

Because x = b, Eq. 19 can be reduced to

$$\frac{a}{\dfrac{v_1^2}{2\,g}} = \frac{5.2}{\dfrac{b}{B_o}} \qquad (20)$$

Eq. 20 may not be directly applicable to the manifold stilling basin, however, because the fundamental theory and investigations leading to Eq. 18 were developed for a single jet of infinite length discharging into a medium of infinite extent; whereas in this study the tailwater surface confines the flow and adjacent jets interfere with each other.

Use of Single Jet Theory.—In the case of a manifold having numerous jets issuing from it with certain boundary conditions imposed, the quantity of fluid surrounding the jet is limited, and the following can be expected to occur:

1. There will be mutual interference between patterns of jet diffusion.
2. The quantity of flow that can be entrained by the jet will be restricted by the boundary conditions.

Both of these factors will result in less efficient diffusion of the jet that, in turn, will result in a greater magnitude of v_{max} and consequently a greater boil height for a given set of conditions. Therefore, an expression similar to Eq. 20 can be written for a manifold with numerous jets and corresponding boundary conditions.

$$\frac{a}{\frac{v_1^2}{2g}} = \frac{C}{\frac{b}{B_o}} \qquad (21)$$

Because the coefficient C reflects the residual kinetic energy, it can be expected to be larger than 5.2.

Limiting Conditions.—Two limiting sets of conditions can be foreseen in which the manifold theory with numerous jets will approach the single jet theory. These are:

1. When the tailwater b is low, or the w/s-values are very small, very little interference will be expected between jet diffusion patterns and, consequently, data obtained under these conditions will approach the horizontal line at unity in Fig. 3.
2. At high tailwater and/or large values of w/s, the manifold behavior will approach single-jet conditions. In this case, however, the manifold will behave as one large jet rotated 90° in the horizontal plane with the cross bar elements serving primarily as vanes to guide the flow in a vertical direction rather than creating a series of individual jets.

An expression is now derived for the limiting case of a manifold acting as a single large jet. The necessary assumptions for this development are:

1. A constant incoming flow Q.
2. The momentum flux m per unit area is the same whether the flow is coming through n_1 number of slots of B_{o_1} width, or through n_2 slots of B_{o_2} width.

Under these conditions $m_1 = m_2$ or

$$\rho n_1 B_{o_1} v_1^2 = \rho n_2 B_{o_2} v_2^2 \qquad (22)$$

and solving for v_2

$$v_2 = \sqrt{\frac{n_1}{n_2}}\sqrt{\frac{B_{o_1}}{B_{o_2}}}\; v_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (23)$$

If $B_{o_2} = 1$ ft and $n_2 = 1$ (representing an opening of 1 sq ft)

$$v_2 = \sqrt{n_1 B_{o_1}}\; v_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (24)$$

Now $v_{max} = v_2$ can be substituted in Eq. 17 for b/B_o-values up to approximately 5.2—or, in which $B_o = 1$ ft, to tailwater depths of approximately 5.2 ft—to yield

$$\frac{a}{\frac{v_1^2}{2\,g}} = \frac{\frac{\left(\sqrt{n_1 B_{o_1}}\, v_1\right)^2}{2\,g}}{\frac{v_1^2}{2\,g}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (25)$$

or

$$\frac{a}{\frac{v_1^2}{2\,g}} = n_1 B_{o_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (26)$$

This expression for the single large jet is a horizontal line when located on a plot similar to Fig. 3, and is approached asymptotically by the curves for the manifold data for intermediate values of w/s and tailwater depth b.

The foregoing theory was tested and many of the desired relationships were obtained as a result of the laboratory investigation.

EXPERIMENTAL EQUIPMENT AND PROCEDURE

In accordance with the foregoing analysis of the problem, a generalized model study of a manifold stilling basin was conducted in the Hydraulics Laboratory of Colorado State University, Fort Collins, Colo. The objectives of the experiments were:

1. To evaluate energy dissipation in a model manifold stilling basin in terms of boil height a and wave height h, at the boundary; and
2. To determine the validity of the general form of Eq. 21 and values of the coefficient C when this equation is applied to a manifold under a given set of conditions.

Fig. 6 shows the general layout of the experimental equipment. The model of the manifold proper was constructed with a length of 8 ft, a width of 1 ft, and a depth of 1 ft at the inlet end. Along the manifold the width remained constant but the depth decreased linearly to zero at the downstream end. Initially, flow was brought horizontally into the manifold through a round-to-square transition. This method gave a non-uniform velocity distribution, however, and was

abandoned in favor of a 1-ft square inlet section 8 ft long and installed on the same slope as the bottom of the manifold—see Fig. 7 where an arrow indicates the direction of flow in square inlet section. With this shape of inlet section the velocity distribution into the manifold was considered satisfactory and flow from the manifold was also reasonably uniform (Fig. 8)

FIG. 7.—MODEL OF MANIFOLD WITH SQUARE AND SLOPING INLET

Square cross bars 1 in., 2 in., and 4 in. across, and round cross bars of in. diameter, were used in studying the effect of cross-bar size, shape, a spacing. Boundary conditions consisted of a simulated channel with sides i stalled on a 2:1 slope and a vertical bulkhead at the upstream end of the man fold.

The equipment and facilities used had the following limitations:

1. The model of the manifold had a fixed length, cross section, and botto slope.

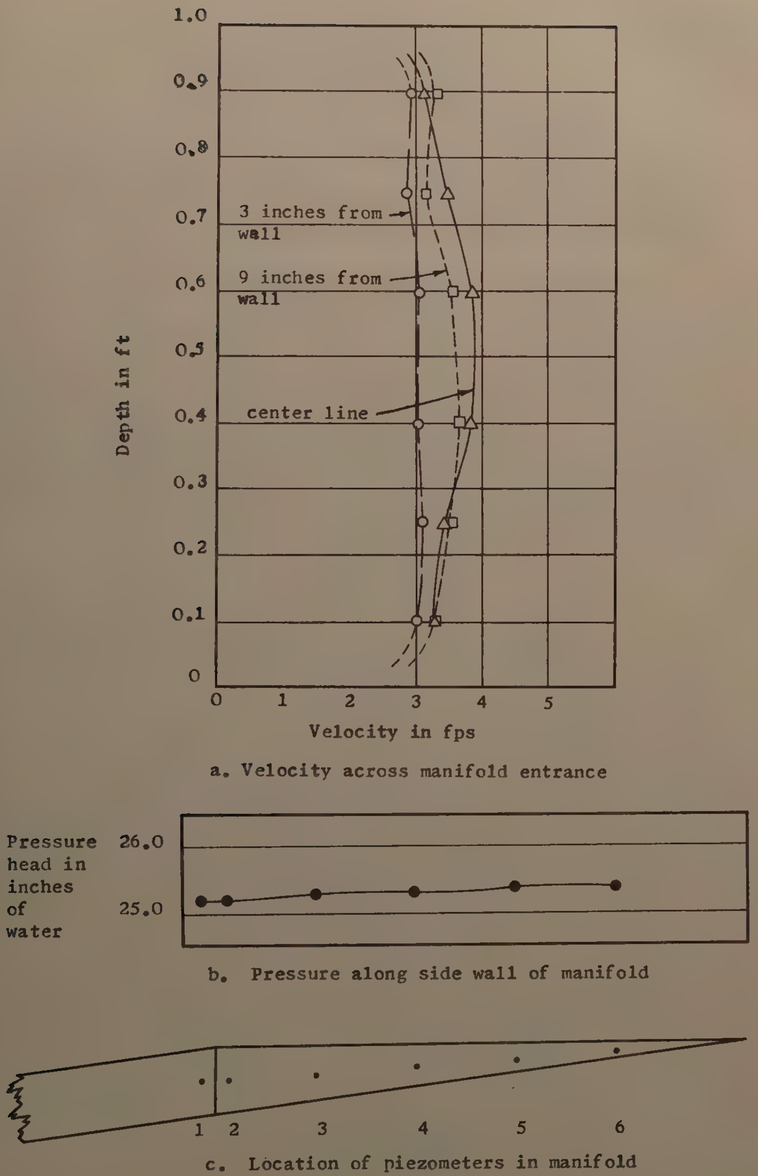

FIG. 8.—DISTRIBUTION OF VELOCITY AND PRESSURE WITH THE SQUARE AND SLOPING INLET. Q = 3.0 cfs, b = 1.5 ft, S = 2 in., w = 2 in.

2. A maximum discharge of 3.5 cfs from the supply pump.
3. A maximum tailwater depth of 1.5 ft due to the depth of the box containing the model.
4. Difficulty in making precise measurements of:
    a. The average elevation of a rough water surface, and
    b. The average wave height at the given boundary wall.

During each experimental run measurements were made of the mean water-surface elevation (or tailwater elevation), velocity head of the jet, height of boil, height of waves, and pressure within the manifold.

The conditions under which this experiment was conducted were:

1. At a discharge of 3 cfs, the cross-bar size, shape, and spacing were varied for several values of tailwater elevation.
2. For several values of discharge, the tailwater elevation was held constant and the cross-bar size, shape, and spacing were varied.

The single discharge was used because preliminary experimentation had shown the ratio v_1/v_0 to be constant for all values of discharge, if the geometry of the manifold and the height of the tailwater were held constant. Further experimentation, however, showed this ratio to vary somewhat under certain conditions. Consequently, for some conditions v_1/v_0 cannot be considered constant, particularly at low tailwater elevations.

RESULTS

The results of systematic laboratory experiments on a model of a manifold stilling basin have been analyzed as follows:

1. Velocity profiles and pressure distributions.
2. Water surface profiles.
3. Analysis of dimensionless plots.

Velocity Profiles and Pressure Distributions.—The first step toward accomplishing these objectives was to obtain flow conditions from the manifold that were as nearly ideal as possible. As previously mentioned, the model was first constructed with a horizontal round-to-square transition immediately upstream from the manifold. This arrangement proved unsatisfactory, however, because both the velocity distribution and pressure distribution inside the manifold were not uniform. The velocity varied considerably in both the horizontal and vertical directions at the inlet end and also in a longitudinal direction along the manifold. Furthermore, the pressure along the inside of the manifold had a steady increase in the downstream direction. These non-uniform flow conditions appeared to be caused by the relatively short round-to-square transition and the angle that the entering flow made with the bottom of the manifold. Both these problems were treated by abandoning the round-to-square transition in favor of a longer, square inlet section installed on the same slope as the bottom of the manifold.

Fig. 8 shows the velocity and pressure distributions obtained with this square and sloping inlet section. The velocity and pressure distributions obtained are considered as nearly ideal as practical for this experiment.

Water Surface Profiles.—Throughout the experiment, water surface profiles were taken longitudinally along the center-line of the manifold and transversally to the manifold one foot distance downstream from the headwall.

With the square cross bars in place, the longitudinal water surface profile above the manifold had the shape of a smooth curve with a descending gradient in the downstream direction along the manifold. The slope of the water surface tended to flatten as the depth of tailwater increased.

With the round cross bars in place, the water surface above the manifold reached a maximum near the downstream end of the manifold. Water appeared to flow around the round cross bars, instead of being turned upward in a vertical direction and, thus, retained a considerable component on velocity in the downstream direction. The performance of the manifold when using round cross bars was considered to be unsatisfactory and further tests with the round cross bars were abandoned.

Fig. 9 shows two water-surface profiles taken transversally to the manifold for square cross bars. The profiles are shown in relation to the manifold and sloping sidewalls. Note that the boil is largely dissipated long before it reaches the banks. A secondary flow is set up by the shear of the jet. This flow carries the water from the dissipated boil to the banks where a reduction of velocity causes a slight rise in elevation of the water surface.

Analysis of Dimensionless Plots.—Although a relatively large number of deimensionless plots were prepared during the analysis of the data, only those having a direct application to the problem of manifold stilling basin design are discussed here. In Fig. 10 the relationship of the velocity of flow at the entrance of the manifold v_o to the initial velocity of the jet v_1 is shown as a function of the parameters b/s, and w/s in which b is the depth of the tailwater; s denotes the size of the manifold crossbars; and w is the width of the spacing between the manifold crossbars.

From Fig. 10 the following observations are made:

1. At relatively high tailwater depths the v_1/v_o-data have a trend toward a vertical asymptote which is shown in Fig. 10.

2. As b/s decreases to $b/s \approx 5$, v_1/v_o becomes smaller. This trend is caused by the marked decrease in tailwater depth along the manifold. The lower hydrostatic head at the downstream end of the manifold then results in a greater discharge from this end and a consequent reduction in flow (and velocity) from the upstream part of the manifold.

3. As the magnitude of w/s increases, v_1/v_o becomes smaller. This appears logical, because it would be expected that the initial jet velocity would become smaller as the width of opening becomes larger with respect to the size of the cross-bar.

4. The data, which plot to the right of the asymptote for w/s = 1.0, are for small values of Q. This inconsistency is attributed to the difficulty in measuring an average velocity of the jet at the small discharges.

5. The function of this plot, in manifold design, is to aid in determining the initial jet velocity v_1, when the entrance velocity v_o, the tailwater depth b, and the geometry of the manifold are known.

Fig. 11 shows the relationship between boil height a, initial velocity of the jet v_1, tailwater depth b, and the average width of the jet B_o, for various w/s-values. Fig. 11 is similar to Fig. 3. It contains three essential elements:

1. The horizontal line at $\dfrac{a}{v_1^2/(2g)} = 1.$

2. The straight line portion on a 1 to 1 slope.

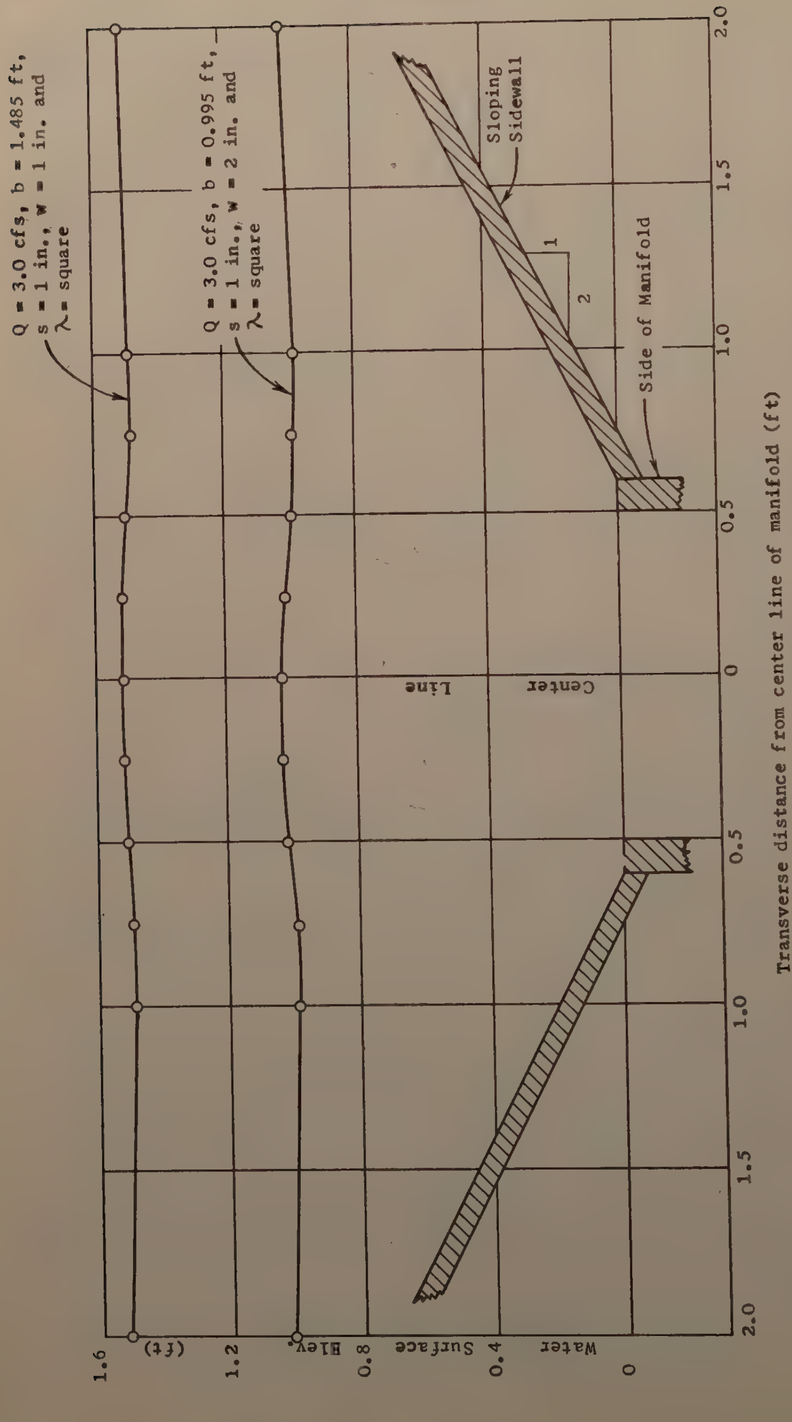

FIG. 9.—TRANSVERSE WATER SURFACE PROFILE AT 1 FT DOWNSTREAM FROM HEADWALL

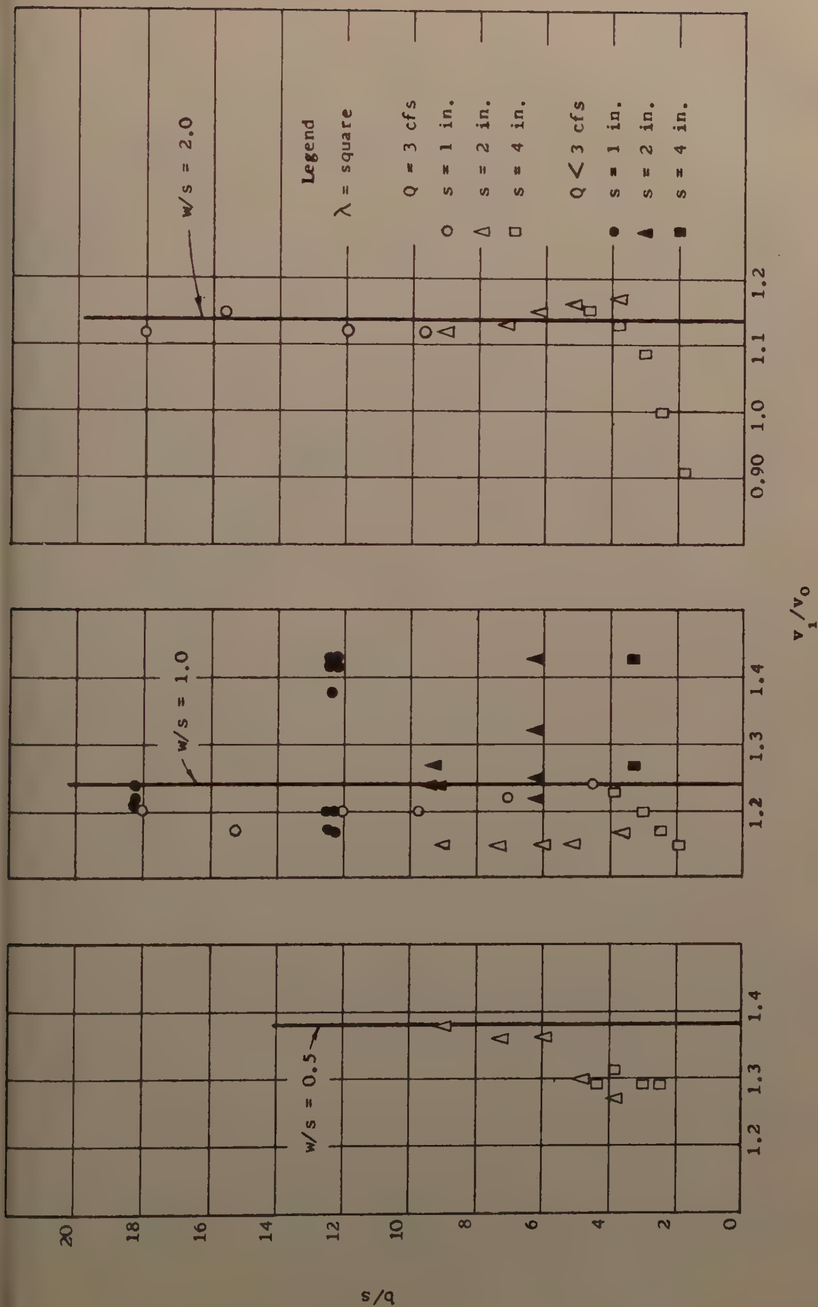

FIG. 10.—VARIATION OF b/s WITH v_1/v_0 AND w/s

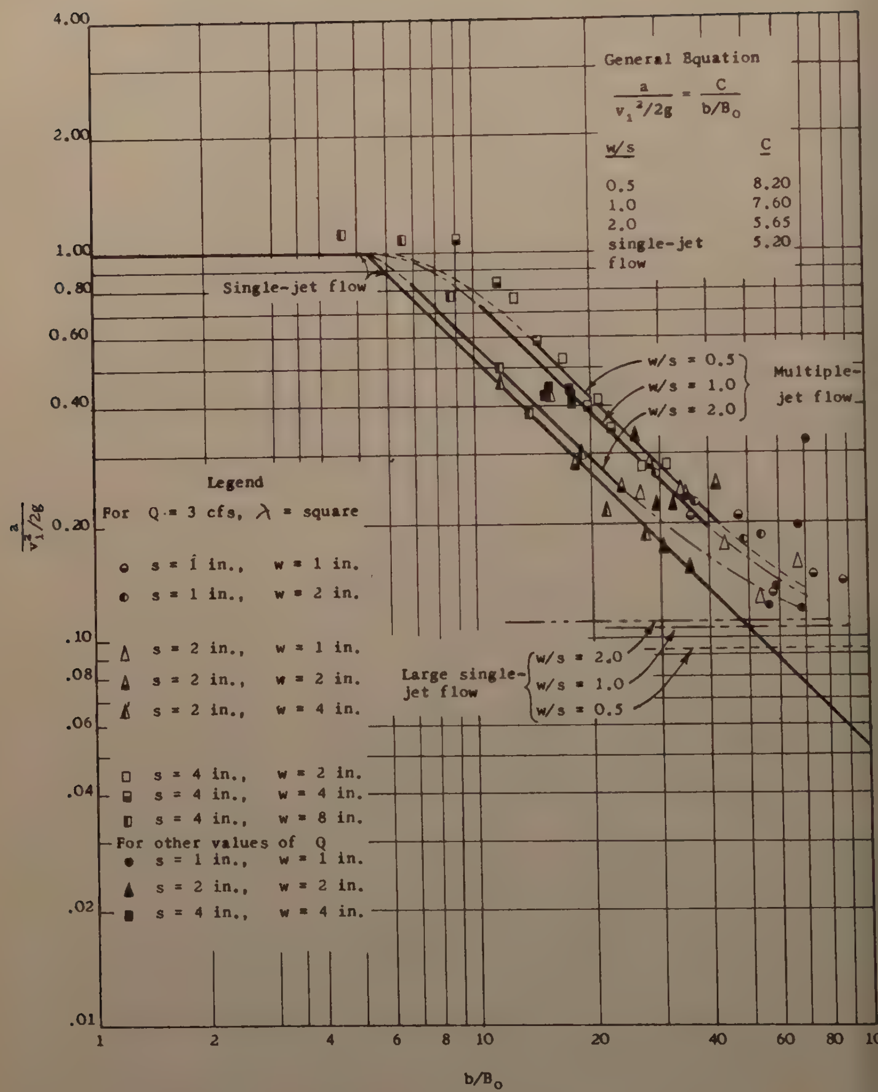

FIG. 11.—VARIATION OF $a/(v_1^2/2g)$ WITH b/B_o AND w/s

3. The horizontal lines near $\frac{a}{v_1^2/(2\,g)} = 0.1.$

The horizontal line near $\frac{a}{v_1^2/(2\,g)} = 1$ represents the limiting condition when the tailwater depth becomes sufficiently low that the boil height a is equal to the initial velocity head $v_1^2/(2\,g)$ of the jet. As the depth increases, the data drift away from the horizontal line toward lines of 1 to 1 slope because the centerline velocity of the jets is being reduced by diffusion of the jet into the deeper tailwater. According to the jet theory[11], this drift away occurs where the tailwater depth is equal to 5.2 times the effective initial jet width B_o.

The straight line portion of the curves on the 1 to 1 slope is significant in that it:

1. Proves the validity of Eq. 21.
2. Establishes the coefficient C in Eq. 21 for specific w/s - values.
3. Supports the theory of less efficient diffusion of jets due to mutual interference. It does this by the fact that these curves indicates a greater boil height, for a given set of conditions, than does the single jet theory.
4. Shows the data tend to drift toward a horizontal asymptote near $\frac{a}{v_1^2/(2\,g)} = 0.1$ for high values of tailwater depth.

The horizontal lines near $\frac{h}{v_1^2/(2\,g)} = 0.1$ show the limiting operating conditions of the manifold as it tends to function as a single large two-dimensional jet with its axis parallel to the manifold. When these operating conditions are reached, the effective width of the jet changes from a fraction of the width of opening w to the entire width B of the manifold and has a velocity v_2. The large single jet is oriented at 90 degrees in the horizontal plane with respect to the small jets.

In Fig. 12 the w/s ratio is plotted against the coefficient for Eq. 21 (multiple-jet flow), as determined from Fig. 11. The equation coefficient 5.2 for the single-jet flow is also plotted against w/s-values for comparative purposes. From Fig. 12 the following observations are made:

1. As the w/s - ratio becomes either large, or small, the behavior of the manifold approaches the single-jet flow. The lower portions of the curve have been determined by judgment because no data are available. As the size s of the manifold crossbar becomes large with respect to the width of the opening w, the jets are separated a sufficient distance to have a negligible effect on one another.
2. The upper portion of the curve has a definite trend toward the single-jet coefficient. It is possible to imagine, however, a condition in which the w/s - ratio is so large that the jets are no longer turned upward in a vertical direction with the result being a poor flow condition.
3. The bulge in the central portion of the curve is caused by mutual interference between the jets of the manifold and occurs between the two regimes described in item 1. From Fig. 12 it appears that part of this experiment was conducted near the point of maximum interference between jets.

Fig. 13 is the same type plot as Fig. 11, the only difference being that the parameter $\frac{h}{v_1^2/(2\,g)}$ is plotted as the ordinate. In this parameter h is the height of the waves at the boundary, and the other terms are identical to those defined in the discussion of Fig. 11. The basic curves (straight lines) are located in the same relative positions as the curves in Fig. 11 for respective values of w/s. The data at the ends of the curves appear to be approaching horizontal asymptotes as the data did in Fig. 11. This is to be expected because a direct relationship exists between wave height and boil height.

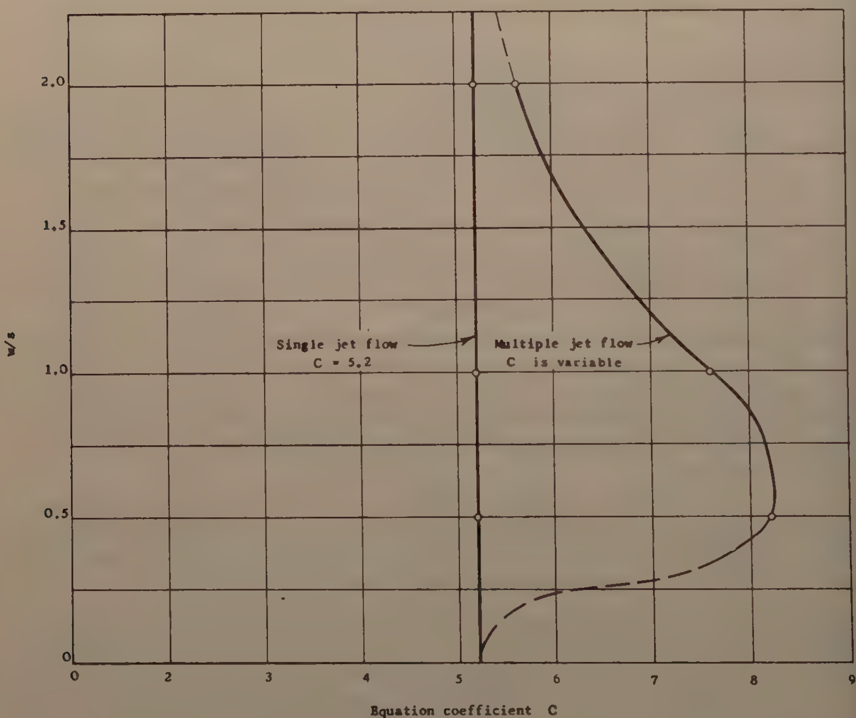

FIG. 12.—COMPARISON OF THE VARIATION OF C WITH w/s FOR SINGLE-JET FLOW AND MULTIPLE-JET FLOW

A primary value of this plot is that the wave height h can be determined, providing the other terms are known. In this way the erosion that takes place for a given wave height can be determined from other data relating erosion to wave height.

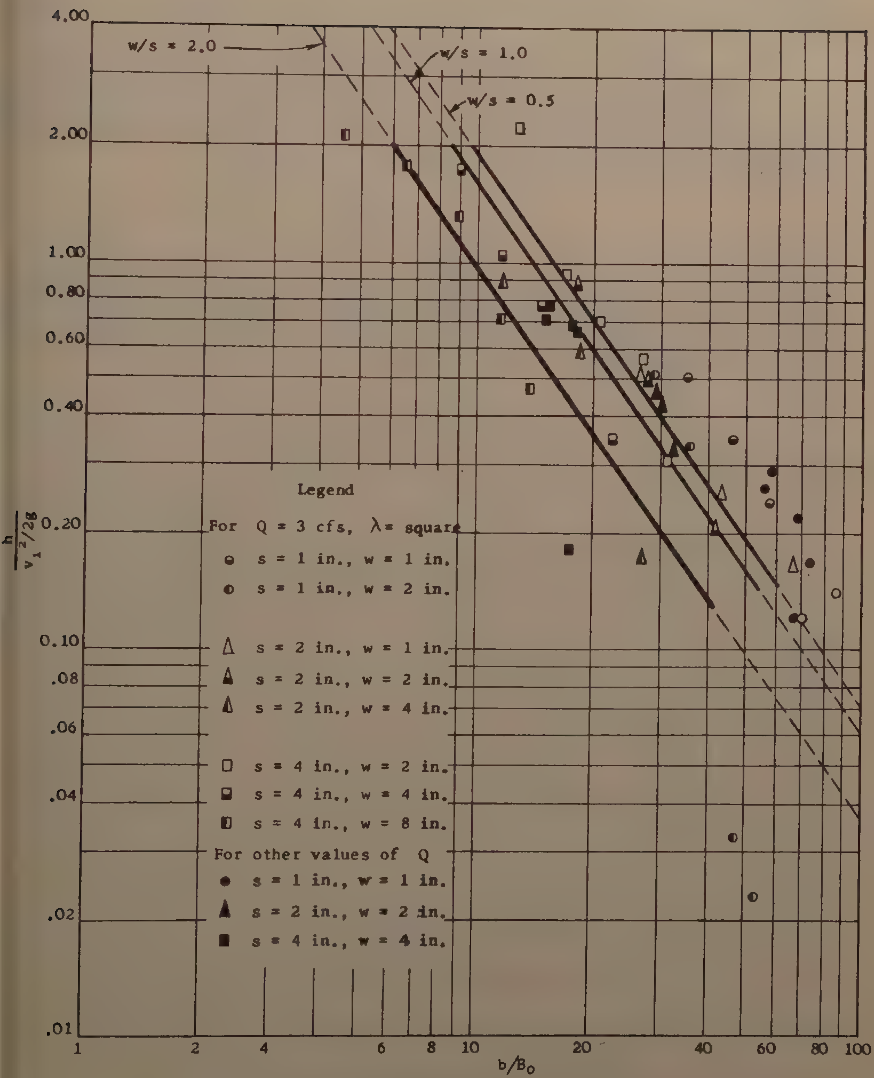

FIG. 13.—VARIATION OF $h/(v_1{}^2/2g)$ WITH b/B_0 AND w/s

The foregoing information can be used to design a manifold stilling basin as follows:

1. The quantity of flow Q and tailwater depth b must be known.
2. The flow condition at the entrance of the manifold is assumed to be uniform, for computation of v_o.
3. The geometry of the manifold is assumed following the criterion $L/\sqrt{A} = 8$ and then checked by the criteria presented in this paper to be sure of proper hydraulic design based on the magnitude of wave height h which can be considered tolerable. The designer may wish to make minor modifications in the design to obtain an economical and practical shape.

EXAMPLE PROBLEM

The following example illustrates the use of the design criteria presented in this paper. The problem is to design a manifold stilling basin for the outlet of a pipe drop so that the wave height (run-up) on the 2:1 side slopes of the channel downstream does not exceed 1.5 ft.

Known:

Discharge Q = 300 cfs, diameter of pipe = 72 in., and minimum tailwater depth b = 8 ft.

Assumed:

1. Uniform distribution of velocity at the entrance of the manifold.
2. For the geometry of the manifold, assume entrance dimensions of 6 ft by 6 ft and $L/\sqrt{A} = 4$. Then $L = 4\sqrt{A} = 4\sqrt{36} = 24$ ft. Also assume, for ease in construction, that w = 12 in. and s = 12 in. which makes the number of slots n = 12 and w/s = 1.0.

Computation:

$$v_o = \frac{Q}{A} = \frac{300}{\frac{\pi\, 6^2}{4}} = 10.6 \text{ fps}$$

$$\frac{b}{s} = \frac{8}{1} = 8$$

From Fig. 10, $v_1/v_o = 1.24$

$$v_1 = 1.24 \times 10.6 = 13.2 \text{ fps}$$

The area A_1 and discharge Q_1 of a single jet is

$A_1 = 6\, B_o$ and

$Q_1 = Q/n = 300/12 = 25$ cfs

Because $A_1 = Q_1/v_1$, the effective width of jet is

$$B_o = \frac{25}{(6)(13.2)} = 0.316 \text{ ft}$$

Then $\dfrac{b}{B_o} = \dfrac{8}{0.316} = 25.3.$

From Fig. 11, $\frac{a}{v_1^2/(2g)} = 0.3$, so that

$$a = 0.3 \frac{(13.2)^2}{64.4} = 0.81 \text{ ft}$$

From Fig. 13, $\frac{h}{v_1^2/2g} = 0.42$, and

$$h = 0.42 \frac{(13.2)^2}{64.4} = 1.14 \text{ ft}$$

which is considerably less than the allowable run-up of 1.5 ft. Therefore, the minimum tailwater could be decreased (by raising the manifold or lowering the downstream control) or the jet velocity could be increased if cost would thereby be reduced.

CONCLUSIONS

1. In general, the dissipation of kinetic energy is caused by shear resistance, pressure resistance, and the turbulence associated with shear and pressure resistance.
2. In some cases, real economies can be effected by dissipating energy in a jet issuing vertically upward.
3. The dissipation of energy in a jet issuing vertically upward has two important advantages:
   a. The jet entrains a part of the surrounding fluid and in so doing it distributes its momentum throughout a greater mass. Furthermore, much of the kinetic energy is converted into heat from the resulting shear, either directly or indirectly, by the creation of relatively fine-grained turbulence.
   b. The kinetic energy that remains in the diffused jet as it reaches the surface of the tailwater causes the jet to rise in a boil above the tailwater. The boil then spreads radially causing a rapid reduction and dispersion of the remaining kinetic energy.
4. The manifold stilling basin has been used as a low-cost structure for combining one stream of high-velocity flow with another stream of flow, both in open channels and in closed conduits.
5. To be suitable as an energy dissipating device in the field of hydraulics, a manifold must have:
   a. A uniform distribution of flow issuing from the manifold along its entire length.
   b. Practical dimensions and shape to keep construction and maintenance costs as small as possible.
6. With the kinetic energy from a manifold being dissipated in a vertical direction, no high-velocity currents, or large concentrations of flow, are directed against either the bed or banks.

7. The design of the inlet section to the manifold is important in obtaining uniform flow from the manifold.

8. Eq. 21, which is based on a single-jet theory with certain assumptions applied, is found to be valid over the anticipated range of conditions. The coefficient C has been determined for specific conditions of geometry and flow.

9. The functioning of the manifold which has been tested was found to approach two limits:

a. The characteristics of a series of single small jets (of width B_o and length B) at low tailwater depths, or the characteristics of a single large jet (of width B and length L) at high tailwater depths.

b. The characteristics of a single jet as the w/s - ratio becomes either large or small.

ACKNOWLEDGMENTS

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APPENDIX - NOTATION

A = cross-sectional area of the manifold at the inlet end;

a = boil height;

B = width of the manifold;

B_o = effective initial width of jet issuing from manifold;

b = tailwater depth;

C = coefficient reflecting residual kinetic energy;

F = Froude number;

g = acceleration of gravity;

H = height of the manifold at the entrance;

h = wave height;

h_1 = height of manifold at a point;

K = a constant;

L = length of manifold;

m = momentum flux per unit area;

n = number of slots;

p = pressure at a point in the manifold;

p_o = pressure at entrance to manifold;

Q = quantity of flow across the section in the manifold;

Q_o = quantity of flow entering the manifold;

R = Reynolds number;

S = distance from the downstream end of the manifold;

s = size of cross bar;

V = mean velocity;

v = velocity at a point in the manifold;

v_1 = initial velocity of jet issuing from manifold;

v_o = mean velocity at entrance to manifold;

v_{max} = centerline velocity in jet at point x;

w = width of opening;

x = distance along the jet;

z = elevation;

$\Delta\gamma$ = difference in specific weight across the air-water interface;

γ = specific weight of water;

λ = shape of cross bar;

μ = dynamic viscosity of water; and

ρ = density of water.

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DRAWDOWN AROUND A PARTIALLY PENETRATING WELL

By Mahdi S. Hantush[1]

SYNOPSIS

Equations of unsteady drawdown around a well partially screened in and steadily discharging from an artesian aquifer of uniform thickness and uniform hydraulic properties are developed. The discharge is supplied by the reduction of storage through expansion of water and the concomitant compression of the aquifer. The solutions are put in forms amenable to relatively simple computation. The results are compared with that of the case of complete penetration. Application of the theory to analysis of aquifer tests will be given in a subsequent paper.

INTRODUCTION

Producing wells frequently do not completely penetrate the aquifer from which they are pumping. The hydraulics of such wells is therefore different from that of wells that fully penetrate the aquifer. The problem of partial penetration has long been recognized, and approximate steady-state solutions for various field conditions have been advanced by J. Kozeny,[2] A. M. ASCE

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1 Senior Hydrol. and Prof. of Hydr., New Mexico Inst. of Mining and Tech., Socorro, N. M. On leave from the College of Engineering, Univ. of Baghdad, Iraq.

2 "Theorie und Berechnung der Brunnen," by J. Kozeny, Wasserkraft u. Wasserwirtschaft, Vol. 28, 1933, p. 101.

M. Muskat,[3] P. Ya. Polubarinova-Kochina,[4] and Hantush and C. E. Jacob,[5] F. ASCE. An exact theory of steady-state flow into a partially penetrating well has been developed by Don Kirkham.[6] The use of these solutions is limited by the fact that they describe the flow under the equilibrium condition only, a condition rarely attained during actual periods of well operation.

The purpose of this paper is to develop unsteady drawdown equations by taking into consideration the length and space positon of the water entry face (screened section) of both the pumped well and the observation well.

Notation.—The symbols in this paper are defined where they first appear. They are assembled alphabetically for convenience, in the Appendix.

DRAWDOWN EQUATIONS

A well whose length of water entry is less than the thickness of the aquifer it penetrates is known as a partially penetrating well. Fig. 1 illustrates conditions characteristic of partially penetrating wells in a confined aquifer. The flow pattern to such wells is three-dimensional rather than the radial (two-dimensional) flow assumed to exist around fully penetrating wells. The drawdown around these wells depends, therefore, among other things, on the space position of the point of observation. Consequently, the drawdown observed in partially penetrating wells will depend on the length and the space position of the screened or perforated sections (water entry portion) of the observation wells.

DRAWDOWN IN PIEZOMETERS

Piezometers are small diameter pipes driven into an aquifer, so that entrance of water into them is solely from the bottom. The drawdown s in a piezometer having a depth of penetration z and being at a distance r from a steadily discharging well that is screened throughout its depth of penetration l, has been obtained by the writer[7] for the case in which the pumped well is of a small radius and the aquifer is of a uniform thickness b, homogeneous, elastic, isotropic, nonleaky, and infinite in areal extent (see Fig. 1 for drawdown equations of this case). A similar analysis will yield a more general solution for the problem if the screen of the pumped well does not extend to the top of the aquifer. In the case of a nonleaky aquifer that is drained by a well of a constant discharge Q and whose screen lies between the depths l and d ($l > d$), the

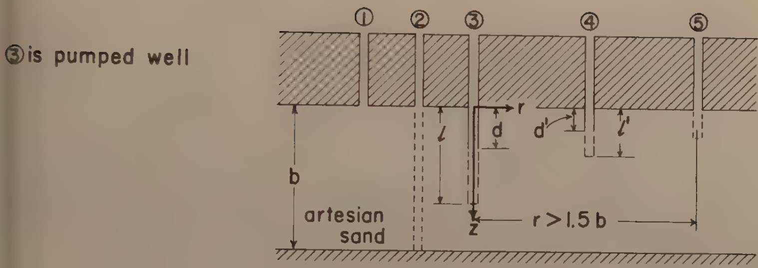
3 "The Flow of Homogeneous Fluids Through Porous Media," by M. Muskat, McGraw-Hill Book Co., Inc., New York, N. Y., 1937; or J. W. Edwards Brothers, Inc., Ann Arbor, Mich., 1946.

4 "Theory of Filtration of Liquids in Porous Media," by P. Ya, Polubarinova-Kochina, Advances in Applied Mechanics, Vol. 2, 1951, p. 207.

5 "Steady Three-Dimensional Flow to a Well in a Two-Layered Aquifer," by M. S. Hantush and C. E. Jacob, Transactions, Amer. Geophysical Union, Vol. 36, 1955, p. 28

6 "Exact Theory of Flow Into a Partially Penetrating Well," by Don Kirkham, Journal of Geophysical Research, Vol. 64, 1959, p. 1317.

7 "Nonsteady Flow to a Well Partially Penetrating an Infinite Leaky Aquifer," by M. S. Hantush, Proceedings, Iraq i Scientific Soc., 1957, pp. 10-19; also reprinted by New Mexico Inst. of Mining and Tech., Socorro, N. M.



Equations of drawdown if $d = d' = 0$

$$u = \frac{r^2 S_s}{4Kt}$$

in: ② & ⑤ $\bar{s} = \frac{Q}{4\pi K b} W(u)$

① & ③ $s = \frac{Q}{4\pi K l}\left[\frac{l}{b} W(u) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi l}{b} W_n\left(u, \frac{n\pi r}{b}\right)\right]$

④ $\bar{s} = \frac{Q}{4\pi K l}\left[\frac{l}{b} W(u) + \frac{2b}{\pi^2 l'}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi l}{b}\sin\frac{n\pi l'}{b} W\left(u, \frac{n\pi r}{b}\right)\right]$

for $b = \infty$ $s(r, z, t) = \frac{Q}{8\pi K l}\int_u^{\infty}\frac{e^{-y}}{y}\left[\mathrm{erf}\left(\frac{l-z}{r}\sqrt{y}\right) + \mathrm{erf}\left(\frac{l+z}{r}\sqrt{y}\right)\right]\cdot dy$

$$W(u) = \int_u^{\infty}\frac{e^{-y}}{y}dy$$

$$W\left(u, \frac{n\pi r}{b}\right) = \int_u^{\infty}\frac{dy}{y}\exp\left[-y - \frac{\left(\frac{n\pi r}{b}\right)^2}{4y}\right]$$

FIG. 1.—DIAGRAMMATIC REPRESENTATION OF WELLS PARTIALLY PENETRATING AN ARTESIAN AQUIFER

solution is given by either of the following two equations (see Fig. 1 for coordinate system):

$$s = \frac{Q}{4\pi K b}\left[W(u) + f\left(u, \frac{r}{b}, \frac{l}{b}, \frac{d}{b}, \frac{z}{b}\right)\right] \qquad (1)$$

or

$$s = \frac{Q}{8\pi K (l - d)}\left[M\left(u, \frac{l + z}{r}\right) + M\left(u, \frac{l - z}{r}\right) + f'\left(u, \frac{b}{r}, \frac{l}{r}, \frac{z}{r}\right) - M\left(u, \frac{d + z}{r}\right) - M\left(u, \frac{d - z}{r}\right) - f'\left(u, \frac{b}{r}, \frac{d}{r}, \frac{z}{r}\right)\right] \qquad (2a)$$

in which

$$f = \frac{2b}{\pi (l - d)}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sin\frac{n\pi l}{b} - \sin\frac{n\pi d}{b}\right)\cos\frac{n\pi z}{b} W\left(u, \frac{n\pi r}{b}\right) \qquad (2b)$$

and

$$f'\left(u, \frac{b}{r}, \frac{x}{r}, \frac{z}{r}\right)\sum_{n=1}^{\infty}\left[M\left(u, \frac{2nb + x + z}{r}\right) - M\left(u, \frac{2nb - x - z}{r}\right) + M\left(u, \frac{2nb + x - z}{r}\right) - M\left(u, \frac{2nb - x + z}{r}\right)\right] \qquad (2c)$$

and where

$$u = \frac{r^2 S_s}{4 K t}, \quad \ldots \ldots \ldots \ldots \ldots \quad (2d)$$

t is the time since pumping started, K is the hydraulic conductivity, S_s is the specific storage (volume of water released from storage in a unit volume of the aquifer under a unit head decline, a dimension L^{-1}), and $W(u,y)$ is[8] what is known as the well function for leaky aquifers. The latter function has been tabulated extensively.

The function $M(u, \beta)$ is defined by the following infinite integral

$$M(u, \beta) = \int_u^\infty \frac{e^{-y}}{y} \operatorname{erf}(\beta\sqrt{y})\, dy \quad \ldots \ldots \ldots \quad (3a)$$

in which erf(x) is the error function. Because $\operatorname{erf}(-x) = -\operatorname{erf}(x)$, it follows that

$$M(u, -\beta) = -M(u, \beta) \quad \ldots \ldots \ldots \ldots \quad (3b)$$

The function $M(u,\beta)$ has[9] been tabulated for a sufficient range of the parameters involved and is given in Table 1. The function can be approximated with sufficient accuracy by

$$M(u, \beta) = 2\left(\sinh^{-1} \beta \quad \frac{2}{\sqrt{\pi}} \beta \sqrt{u}\right), \text{ if } u < \frac{0.05}{2} < .01 \quad \ldots \quad (4)$$

$$M(u, \beta) = 2\left(\sinh^{-1} \beta - \beta \operatorname{erf}(\sqrt{u})\right), \text{ if } u < \frac{.05}{\beta^2} \quad \ldots \ldots \quad (5)$$

and

$$M(u, \beta) = W(u) \qquad , \text{ if } u > \frac{5}{\beta^2} \quad \ldots \ldots \quad (6)$$

in which $\sinh^{-1} \beta$ is the inverse hyperbolic sine of β and W(u) is the well function for nonleaky aquifers, or what in the mathematical literature is known as the negative exponential integral of (-u). This function is available in tabular form.[10]

Equation of Drawdown for Relatively Small Values of Time.—By virtue of Eq. 6, f' terms of Eq. 2a can be safely neglected if $u > 5\left[r/(2b-1-z)\right]^2$. For b equals infinity these f' terms vanish also. Thus, if either $t < \left[(2b-1-z)^2 S_s/(20\,K)\right]$; that is for relatively short period of pumping, or the aquifer is

8 "Preliminary Quantitative Study of the Roswell Ground-Water Reservoir," by M. S. Hantush, New Mexico Inst. of Mining and Tech., 1955 (reprinted, 1957); also "Analysis of Data from Pumping Tests in Leaky Aquifers," by M. S. Hantush, Transactions, Amer. Geophysical Union, Vol. 37, 1956, p. 702.

9 Professional Paper 102, Research Div., New Mexico Inst. of Mining and Tech., Socorro, N. M.

10 "Methods for Determining Permeability of Water-Bearing Materials," by L. K. Wenzel, U. S. Geol. Survey, Water-Supply Paper No. 887, 1942, p. 88, also "Hydrology," by C. O. Wisler and E. F. Brater, John Wiley and Sons, New York, N. Y., 1951.

TABLE 1.—VALUES OF THE FUNCTION $M(u,\beta) = \int_u^\infty \frac{e^{-y}}{y} \operatorname{erf}(\beta\sqrt{y})\, dy$

u	β →	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.2	1.4	1.6	1.8	2.0
	0	0.1997	0.3974	0.5913	0.7801	0.9624	1.1376	1.3053	1.4653	1.6177	1.7627	2.0319	2.2759	2.4979	2.7009	2.8872
	1	0.1994	0.3969	0.5907	0.7792	0.9613	1.1363	1.3037	1.4635	1.6157	1.7605	2.0292	2.2728	2.4943	2.6968	2.8827
	2	0.1993	0.3967	0.5904	0.7788	0.9608	1.1357	1.3031	1.4628	1.6148	1.7595	2.0281	2.2715	2.4929	2.6951	2.8809
	3	0.1993	0.3966	0.5902	0.7785	0.9605	1.1353	1.3026	1.4622	1.6142	1.7588	2.0272	2.2705	2.4917	2.6938	2.8794
	4	0.1992	0.3965	0.5900	0.7783	0.9602	1.1349	1.3022	1.4617	1.6137	1.7582	2.0265	2.2696	2.4907	2.6927	2.8782
10^{-6}	5	0.1992	0.3964	0.5898	0.7780	0.9599	1.1346	1.3018	1.4613	1.6132	1.7577	2.0259	2.2689	2.4899	2.6918	2.8772
	6	0.1991	0.3963	0.5897	0.7779	0.9596	1.1343	1.3014	1.4609	1.6127	1.7572	2.0253	2.2682	2.4891	2.6909	2.8762
	7	0.1991	0.3962	0.5895	0.7777	0.9594	1.1341	1.3011	1.4605	1.6123	1.7568	2.0248	2.2676	2.4884	2.6901	2.8753
	8	0.1990	0.3961	0.5894	0.7775	0.9592	1.1338	1.3009	1.4602	1.6120	1.7563	2.0243	2.2670	2.4877	2.6894	2.8745
	9	0.1990	0.3960	0.5893	0.7774	0.9590	1.1336	1.3006	1.4599	1.6116	1.7560	2.0238	2.2665	2.4871	2.6887	2.8737
	1	0.1989	0.3959	0.5892	0.7772	0.9588	1.1334	1.3003	1.4596	1.6113	1.7556	2.0234	2.2660	2.4865	2.6880	2.8730
	2	0.1987	0.3954	0.5883	0.7760	0.9574	1.1316	1.2983	1.4572	1.6086	1.7526	2.0198	2.2618	2.4818	2.6827	2.8671
	3	0.1984	0.3949	0.5876	0.7751	0.9562	1.1302	1.2967	1.4554	1.6066	1.7504	2.0171	2.2587	2.4782	2.6786	2.8625
	4	0.1982	0.3945	0.5871	0.7744	0.9553	1.1291	1.2953	1.4539	1.6049	1.7485	2.0148	2.2560	2.4751	2.6752	2.8587
10^{-5}	5	0.1981	0.3942	0.5866	0.7737	0.9544	1.1281	1.2941	1.4526	1.6034	1.7468	2.0128	2.2536	2.4724	2.6721	2.8553
	6	0.1979	0.3939	0.5861	0.7731	0.9537	1.1271	1.2931	1.4513	1.6020	1.7452	2.0110	2.2515	2.4700	2.6694	2.8523
	7	0.1978	0.3936	0.5857	0.7725	0.9530	1.1263	1.2921	1.4502	1.6007	1.7438	2.0093	2.2495	2.4677	2.6669	2.8495
	8	0.1976	0.3933	0.5853	0.7720	0.9523	1.1255	1.2912	1.4492	1.5996	1.7425	2.0077	2.2477	2.4657	2.6645	2.8469
	9	0.1975	0.3931	0.5849	0.7715	0.9517	1.1248	1.2903	1.4482	1.5984	1.7413	2.0062	2.2460	2.4637	2.6623	2.8444
	1	0.1974	0.3929	0.5846	0.7710	0.9511	1.1241	1.2895	1.4473	1.5974	1.7402	2.0049	2.2444	2.4619	2.6603	2.8421
	2	0.1965	0.3910	0.5818	0.7673	0.9465	1.1185	1.2830	1.4398	1.5890	1.7308	1.9936	2.2313	2.4469	2.6434	2.8234
	3	0.1958	0.3896	0.5796	0.7644	0.9429	1.1142	1.2780	1.4341	1.5825	1.7236	1.9850	2.2212	2.4354	2.6305	2.8091
	4	0.1952	0.3883	0.5778	0.7620	0.9398	1.1106	1.2737	1.4292	1.5771	1.7176	1.9778	2.2128	2.4258	2.6197	2.7970
10^{-4}	5	0.1946	0.3873	0.5762	0.7599	0.9372	1.1074	1.2700	1.4250	1.5723	1.7123	1.9714	2.2053	2.4172	2.6101	2.7864
	6	0.1941	0.3863	0.5748	0.7580	0.9348	1.1045	1.2666	1.4211	1.5680	1.7075	1.9656	2.1986	2.4095	2.6014	2.7768
	7	0.1937	0.3854	0.5734	0.7562	0.9326	1.1018	1.2635	1.4176	1.5640	1.7030	1.9603	2.1924	2.4025	2.5934	2.7679
	8	0.1933	0.3846	0.5722	0.7545	0.9305	1.0994	1.2607	1.4143	1.5603	1.6989	1.9554	2.1866	2.3959	2.5860	2.7597
	9	0.1929	0.3838	0.5710	0.7530	0.9286	1.0970	1.2579	1.4112	1.5568	1.6951	1.9507	2.1812	2.3897	2.5791	2.7519
	1	0.1925	0.3831	0.5699	0.7515	0.9267	1.0948	1.2554	1.4083	1.5535	1.6914	1.9463	2.1761	2.3838	2.5725	2.7446
	2	0.1896	0.3772	0.5611	0.7397	0.9120	1.0771	1.2347	1.3846	1.5270	1.6619	1.9109	2.1348	2.3367	2.5195	2.6857
	3	0.1873	0.3727	0.5543	0.7307	0.9007	1.0636	1.2189	1.3666	1.5066	1.6393	1.8838	2.1032	2.3006	2.4788	2.6406
	4	0.1854	0.3689	0.5486	0.7231	0.8912	1.0521	1.2056	1.3513	1.4895	1.6203	1.8610	2.0766	2.2702	2.4447	2.6027
10^{-3}	5	0.1837	0.3655	0.5435	0.7163	0.8828	1.0421	1.1938	1.3379	1.4744	1.6035	1.8409	2.0532	2.2434	2.4146	2.5693
	6	0.1822	0.3625	0.5390	0.7103	0.8752	1.0330	1.1832	1.3258	1.4608	1.5884	1.8228	2.0320	2.2193	2.3875	2.5393
	7	0.1808	0.3597	0.5348	0.7047	0.8682	1.0246	1.1735	1.3147	1.4483	1.5745	1.8061	2.0126	2.1972	2.3626	2.5117
	8	0.1795	0.3571	0.5310	0.6995	0.8618	1.0169	1.1645	1.3044	1.4367	1.5616	1.7907	1.9946	2.1766	2.3395	2.4861
	9	0.1783	0.3547	0.5273	0.6947	0.8557	1.0096	1.1560	1.2947	1.4258	1.5495	1.7762	1.9777	2.1573	2.3179	2.4620
	1	0.1772	0.3524	0.5239	0.6901	0.8500	1.0027	1.1480	1.2855	1.4155	1.5381	1.7625	1.9617	2.1391	2.2975	2.4394
	2	0.1680	0.3340	0.4962	0.6533	0.8040	0.9476	1.0836	1.2121	1.3329	1.4464	1.6527	1.8340	1.9935	2.1342	2.2587
	3	0.1610	0.3200	0.4753	0.6253	0.7691	0.9057	1.0349	1.1564	1.2703	1.3770	1.5697	1.7376	1.8839	2.0116	2.1233
	4	0.1551	0.3083	0.4578	0.6020	0.7400	0.8708	0.9942	1.1100	1.2183	1.3193	1.5008	1.6577	1.7932	1.9103	2.0117
10^{-2}	5	0.1500	0.2981	0.4425	0.5817	0.7146	0.8404	0.9588	1.0696	1.1730	1.2691	1.4410	1.5884	1.7147	1.8229	1.9156
	6	0.1455	0.2890	0.4289	0.5635	0.6919	0.8132	0.9272	1.0336	1.1326	1.2243	1.3877	1.5268	1.6450	1.7454	1.8307
	7	0.1413	0.2807	0.4164	0.5470	0.6713	0.7885	0.8994	1.0008	1.0958	1.1837	1.3394	1.4711	1.5821	1.6756	1.7543
	8	0.1375	0.2731	0.4050	0.5317	0.6522	0.7658	0.8720	0.9707	1.0621	1.1464	1.2951	1.4200	1.5246	1.6120	1.6848
	9	0.1339	0.2660	0.3943	0.5176	0.6346	0.7447	0.8474	0.9428	1.0308	1.1118	1.2541	1.3729	1.4716	1.5534	1.6210
	1	0.1306	0.2593	0.3844	0.5043	0.6181	0.7249	0.8245	0.9167	1.0016	1.0795	1.2159	1.3290	1.4223	1.4991	1.5619
	2	0.1051	0.2084	0.3081	0.4030	0.4920	0.5744	0.6500	0.7186	0.7806	0.8362	0.9297	1.0029	1.0595	1.1026	1.1352
	3	8.74(-2)	0.1731	0.2554	0.3331	0.4053	0.4713	0.5309	0.5842	0.6313	0.6727	0.7400	0.7899	0.8261	0.8519	0.8699
	4	7.39(-2)	0.1462	0.2153	0.2801	0.3397	0.3935	0.4415	0.4837	0.5203	0.5519	0.6015	0.6363	0.6602	0.6760	0.6863
10^{-1}	5	6.32(-2)	0.1248	0.1835	0.2381	0.2878	0.3323	0.3714	0.4052	0.4341	0.4584	0.4955	0.5203	0.5362	0.5462	0.5521
	6	5.44(-2)	0.1074	0.1575	0.2039	0.2458	0.2828	0.3149	0.3423	0.3652	0.3842	0.4122	0.4300	0.4408	0.4471	0.4506
	7	4.71(-2)	9.29(-2)	0.1360	0.1756	0.2111	0.2421	0.2686	0.2909	0.3093	0.3242	0.3455	0.3583	0.3657	0.3698	0.3719
	8	4.10(-2)	8.06(-2)	0.1179	0.1519	0.1821	0.2082	0.2302	0.2484	0.2632	0.2750	0.2913	0.3007	0.3058	0.3084	0.3096
	9	3.67(-2)	7.03(-2)	0.1026	0.1319	0.1576	0.1797	0.1980	0.2130	0.2250	0.2343	0.2468	0.2537	0.2572	0.2589	0.2597
	1	3.13(-2)	6.14(-2)	8.95(-2)	0.1148	0.1369	0.1555	0.1709	0.1833	0.1929	0.2004	0.2101	0.2151	0.2175	0.2186	0.2191
	2	9.01(-3)	1.75(-2)	2.51(-2)	3.16(-2)	3.67(-2)	4.07(-2)	4.35(-2)	4.55(-2)	4.69(-2)	4.77(-2)	4.85(-2)	4.88(-2)			4.88(-2)
	3	2.82(-3)	5.44(-3)	7.68(-3)	9.47(-3)	1.08(-2)	1.17(-2)	1.23(-2)	1.26(-2)	1.28(-2)	1.30(-2)					1.30(-2)
	4	9.20(-4)	1.76(-3)	2.44(-3)	2.96(-3)	3.31(-3)	3.53(-3)	3.66(-3)	3.72(-3)	3.76(-3)	3.77(-3)					3.77(-3)
1	5	3.07(-4)	5.80(-4)	7.96(-4)	9.49(-4)	1.05(-3)	1.10(-3)	1.13(-3)	1.14(-3)	1.15(-3)						1.15(-3)
	6	1.04(-4)	1.95(-4)	2.64(-4)	3.10(-4)	3.38(-4)	3.50(-4)	3.56(-4)	3.59(-4)	3.60(-4)						3.60(-4)
	7	3.56(-5)	6.61(-5)	8.84(-5)	1.02(-4)	1.10(-4)	1.13(-4)	1.15(-4)	1.15(-4)	1.15(-4)						1.15(-4)
	8	1.23(-5)	2.26(-5)	2.99(-5)	3.42(-5)	3.63(-5)	3.72(-5)	3.75(-5)	3.76(-5)	3.77(-5)						3.77(-5)
	9	4.28(-6)	7.79(-6)	1.02(-5)	1.15(-5)	1.21(-5)	1.23(-5)	1.24(-5)	1.24(-5)							1.24(-5)
	10	1.49(-6)	2.70(-6)	3.48(-6)	3.90(-6)	4.07(-6)	4.13(-6)	4.15(-6)	4.16(-6)							4.16(-6)

Values of $M(u,\beta)$ are equal to $W(u)$ for u greater than 10 and all values of β

The numbers in parenthesis are powers of 10 by which the other numbers are raised e.g., 8.74(−2) = 0.087

TABLE 1.–*CONTINUED*

u		β = 2.0	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8
	0	2.8872	3.0593	3.2188	3.3675	3.5064	3.6369	3.7597	3.8757	3.9856	4.0900	4.1894	4.2842	4.3748	4.4616	4.5448	4.6248	4.7018	4.7760	4.8475	4.9167
10^{-6}	1	2.8827	3.0543	3.2134	3.3616	3.5001	3.6301	3.7525	3.8681	3.9775	4.0815	4.1804	4.2747	4.3649	4.4512	4.5340	4.6136	4.6901	4.7638	4.8349	4.9036
	2	2.8809	3.0523	3.2112	3.3592	3.4975	3.6273	3.7495	3.8649	3.9742	4.0779	4.1766	4.2708	4.3608	4.4469	4.5295	4.6089	4.6852	4.7588	4.8297	4.8982
	3	2.8794	3.0507	3.2095	3.3573	3.4955	3.6251	3.7472	3.8625	3.9716	4.0752	4.1738	4.2678	4.3576	4.4436	4.5261	4.6053	4.6815	4.7549	4.8257	4.8940
	4	2.8782	3.0494	3.2080	3.3557	3.4938	3.6233	3.7453	3.8604	3.9694	4.0729	4.1714	4.2653	4.3550	4.4408	4.5232	4.6023	4.6784	4.7516	4.8223	4.8905
	5	2.8772	3.0482	3.2067	3.3544	3.4923	3.6217	3.7436	3.8586	3.9675	4.0709	4.1692	4.2630	4.3526	4.4384	4.5206	4.5996	4.6756	4.7488	4.8193	4.8874
	6	2.8762	3.0471	3.2056	3.3531	3.4910	3.6203	3.7420	3.8569	3.9658	4.0690	4.1673	4.2610	4.3505	4.4362	4.5183	4.5972	4.6731	4.7462	4.8166	4.8846
	7	2.8753	3.0462	3.2045	3.3519	3.4897	3.6190	3.7406	3.8554	3.9642	4.0674	4.1655	4.2591	4.3486	4.4341	4.5162	4.5950	4.6708	4.7438	4.8141	4.8821
	8	2.8745	3.0453	3.2035	3.3509	3.4886	3.6177	3.7393	3.8540	3.9627	4.0658	4.1639	4.2574	4.3467	4.4323	4.5142	4.5929	4.6686	4.7415	4.8118	4.8797
	9	2.8737	3.0444	3.2026	3.3499	3.4875	3.6166	3.7380	3.8527	3.9613	4.0643	4.1623	4.2558	4.3450	4.4305	4.5124	4.5910	4.6666	4.7394	4.8097	4.8774
10^{-5}	1	2.8730	3.0436	3.2017	3.3489	3.4865	3.6155	3.7369	3.8515	3.9600	4.0629	4.1609	4.2542	4.3434	4.4288	4.5106	4.5892	4.6647	4.7375	4.8076	4.8753
	2	2.8671	3.0371	3.1946	3.3412	3.4782	3.6066	3.7274	3.8414	3.9493	4.0517	4.1490	4.2418	4.3304	4.4152	4.4964	4.5744	4.6494	4.7215	4.7911	4.8582
	3	2.8625	3.0321	3.1892	3.3353	3.4718	3.5998	3.7202	3.8337	3.9412	4.0431	4.1400	4.2323	4.3204	4.4048	4.4855	4.5631	4.6376	4.7093	4.7784	4.8450
	4	2.8587	3.0279	3.1846	3.3304	3.4665	3.5941	3.7140	3.8272	3.9343	4.0358	4.1323	4.2243	4.3120	4.3960	4.4764	4.5535	4.6276	4.6989	4.7677	4.8339
	5	2.8553	3.0242	3.1806	3.3260	3.4618	3.5890	3.7087	3.8215	3.9282	4.0294	4.1256	4.2172	4.3046	4.3882	4.4683	4.5451	4.6189	4.6899	4.7582	4.8242
	6	2.8523	3.0209	3.1769	3.3220	3.4575	3.5844	3.7038	3.8163	3.9227	4.0236	4.1195	4.2108	4.2979	4.3812	4.4610	4.5375	4.6110	4.6816	4.7497	4.8153
	7	2.8495	3.0178	3.1735	3.3184	3.4536	3.5802	3.6993	3.8116	3.9177	4.0183	4.1139	4.2049	4.2918	4.3748	4.4543	4.5305	4.6037	4.6741	4.7419	4.8072
	8	2.8469	3.0149	3.1704	3.3150	3.4499	3.5763	3.6951	3.8071	3.9130	4.0133	4.1087	4.1994	4.2860	4.3688	4.4480	4.5240	4.5969	4.6670	4.7346	4.7997
	9	2.8444	3.0122	3.1675	3.3118	3.4465	3.5727	3.6912	3.8030	3.9086	4.0087	4.1038	4.1943	4.2806	4.3632	4.4421	4.5178	4.5905	4.6604	4.7277	4.7926
10^{-4}	1	2.8421	3.0097	3.1647	3.3088	3.4433	3.5692	3.6875	3.7990	3.9044	4.0043	4.0992	4.1894	4.2756	4.3578	4.4365	4.5121	4.5845	4.6542	4.7212	4.7859
	2	2.8234	2.9891	3.1423	3.2845	3.4171	3.5412	3.6576	3.7673	3.8708	3.9688	4.0618	4.1502	4.2345	4.3149	4.3918	4.4654	4.5360	4.6038	4.6690	4.7317
	3	2.8091	2.9733	3.1251	3.2659	3.3970	3.5197	3.6347	3.7429	3.8450	3.9416	4.0332	4.1202	4.2030	4.2820	4.3574	4.4296	4.4988	4.5652	4.6289	4.6903
	4	2.7970	2.9600	3.1106	3.2502	3.3801	3.5015	3.6154	3.7224	3.8233	3.9187	4.0090	4.0948	4.1764	4.2542	4.3285	4.3995	4.4674	4.5326	4.5952	4.6553
	5	2.7864	2.9483	3.0978	3.2363	3.3652	3.4856	3.5984	3.7043	3.8042	3.8985	3.9878	4.0725	4.1531	4.2298	4.3030	4.3729	4.4399	4.5040	4.5655	4.6246
	6	2.7768	2.9378	3.0863	3.2238	3.3518	3.4712	3.5830	3.6880	3.7869	3.8802	3.9686	4.0524	4.1320	4.2077	4.2800	4.3490	4.4150	4.4781	4.5387	4.5969
	7	2.7679	2.9280	3.0757	3.2123	3.3394	3.4579	3.5688	3.6730	3.7710	3.8635	3.9509	4.0338	4.1126	4.1875	4.2588	4.3270	4.3921	4.4544	4.5141	4.5714
	8	2.7597	2.9190	3.0658	3.2017	3.3279	3.4456	3.5557	3.6590	3.7562	3.8479	3.9345	4.0166	4.0945	4.1686	4.2392	4.3065	4.3708	4.4323	4.4912	4.5477
	9	2.7519	2.9105	3.0565	3.1916	3.3171	3.4340	3.5434	3.6459	3.7423	3.8332	3.9191	4.0004	4.0778	4.1509	4.2207	4.2873	4.3508	4.4116	4.4697	4.5255
10^{-3}	1	2.7446	2.9024	3.0478	3.1821	3.3069	3.4231	3.5317	3.6335	3.7292	3.8194	3.9046	3.9852	4.0616	4.1342	4.2033	4.2691	4.3320	4.3920	4.4494	4.5045
	2	2.6857	2.8377	2.9771	3.1056	3.2245	3.3349	3.4377	3.5337	3.6236	3.7080	3.7874	3.8623	3.9329	3.9998	4.0632	4.1233	4.1805	4.2349	4.2867	4.3360
	3	2.6406	2.7881	2.9231	3.0471	3.1616	3.2675	3.3659	3.4575	3.5430	3.6230	3.6981	3.7686	3.8350	3.8975	3.9566	4.0125	4.0654	4.1156	4.1632	4.2084
	4	2.6027	2.7464	2.8776	2.9980	3.1087	3.2110	3.3056	3.3936	3.4754	3.5518	3.6233	3.6902	3.7530	3.8120	3.8676	3.9199	3.9694	4.0161	4.0602	4.1020
	5	2.5693	2.7098	2.8377	2.9548	3.0623	3.1613	3.2528	3.3375	3.4162	3.4894	3.5577	3.6215	3.6813	3.7372	3.7898	3.8391	3.8856	3.9293	3.9705	4.0094
	6	2.5393	2.6767	2.8018	2.9159	3.0205	3.1166	3.2052	3.2871	3.3629	3.4334	3.4989	3.5599	3.6169	3.6702	3.7200	3.7667	3.8105	3.8517	3.8903	3.9267
	7	2.5117	2.6464	2.7688	2.8802	2.9822	3.0757	3.1616	3.2409	3.3142	3.3821	3.4451	3.5036	3.5581	3.6090	3.6564	3.7007	3.7422	3.7810	3.8174	3.8515
	8	2.4861	2.6183	2.7382	2.8472	2.9466	3.0377	3.1213	3.1982	3.2691	3.3346	3.3953	3.4516	3.5038	3.5524	3.5977	3.6398	3.6792	3.7159	3.7502	3.7823
	9	2.4620	2.5920	2.7095	2.8162	2.9134	3.0022	3.0835	3.1582	3.2270	3.2903	3.3489	3.4030	3.4532	3.4998	3.5430	3.5832	3.6206	3.6554	3.6878	3.7181
10^{-2}	1	2.4394	2.5671	2.6825	2.7870	2.8820	2.9687	3.0480	3.1206	3.1873	3.2487	3.3052	3.3574	3.4057	3.4503	3.4917	3.5300	3.5656	3.5987	3.6294	3.6580
	2	2.2587	2.3692	2.4676	2.5552	2.6337	2.7041	2.7673	2.8243	2.8756	2.9218	2.9637	3.0015	3.0357	3.0665	3.0946	3.1200	3.1430	3.1638	3.1827	3.1998
	3	2.1233	2.2212	2.3072	2.3830	2.4496	2.5088	2.5610	2.6073	2.6482	2.6846	2.7168	2.7453	2.7707	2.7932	2.8131	2.8307	2.8464	2.8602	2.8724	2.8832
	4	2.0117	2.0996	2.1759	2.2423	2.3000	2.3503	2.3942	2.4324	2.4658	2.4949	2.5202	2.5423	2.5615	2.5782	2.5927	2.6052	2.6161	2.6256	2.6337	2.6408
	5	1.9156	1.9951	2.0634	2.1221	2.1726	2.2158	2.2531	2.2851	2.3125	2.3361	2.3563	2.3735	2.3883	2.4009	2.4116	2.4207	2.4284	2.4350	2.4405	2.4452
	6	1.8307	1.9031	1.9645	2.0167	2.0610	2.0986	2.1305	2.1574	2.1802	2.1995	2.2157	2.2294	2.2408	2.2504	2.2584	2.2651	2.2706	2.2752	2.2790	2.2821
	7	1.7543	1.8205	1.8760	1.9227	1.9618	1.9946	2.0220	2.0449	2.0640	2.0798	2.0929	2.1038	2.1127	2.1201	2.1261	2.1310	2.1350	2.1382	2.1408	2.1429
	8	1.6848	1.7456	1.7959	1.8378	1.8725	1.9012	[illegible]	1.9444	1.9604	1.9734	1.9841	1.9928	1.9998	2.0055	2.0101	2.0137	2.0166	2.0189	2.0207	2.0221
	9	1.6210	1.6767	1.7236	1.7603	1.7912	1.8164	1.8370	1.8536	1.8671	1.8780	1.8867	1.8937	1.8992	1.9036	1.9071	1.9098	1.9119	1.9135	1.9148	1.9158
10^{-1}	1	1.5619	1.6133	1.6552	1.6892	1.7167	1.7389	1.7568	1.7711	1.7825	1.7915	1.7987	1.8043	1.8087	1.8121	1.8147	1.8168	1.8183	1.8195	1.8204	1.8211
	2	1.1352	1.1596	1.1777	1.1909	1.2004	1.2073	1.2122	1.2156	1.2179	1.2195	1.2206	1.2213	1.2218	1.2221	1.2223	1.2224	1.2225	1.2226	1.2226	1.2226
	3	0.8699	0.8823	0.8907	0.8962	0.8998	0.9021	0.9035	0.9044	0.9049	0.9053	0.9054	0.9056	0.9056	0.9056	0.9057					
	4	0.6863	0.6928	0.6968	0.6992	0.7006	0.7014	0.7019	0.7021	0.7023	0.7023	0.7024									
	5	0.5521	0.5556	0.5576	0.5587	0.5592	0.5595	0.5597	0.5597	0.5597											
	6	0.4506	0.4525	0.4535	0.4540	0.4542	0.4543	0.4543													
	7	0.3719	0.3729	0.3734	0.3736	0.3737	0.3737														
	8	0.3096	0.3102	0.3104	0.3106																
	9	0.2597	0.2600	0.2601	0.2602																
	1	0.2191	0.2193	0.2194																	

Values of $M(u,\beta)$ are equal to $W(u)$ for u greater than 1 and all values of β

u		β = 6.0	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0	8.2	8.4	8.6	8.8	9.0	9.2	9.4	9.6	9.8
	0	4.9835	5.0482	5.1109	5.1718	5.2308	5.2882	5.3440	5.3983	5.4511	5.5026	5.5529	5.6019	5.6497	5.6965	5.7421	5.7868	5.8305	5.8733	5.9151	5.9562
10^{-6}	1	4.9700	5.0343	5.0965	5.1569	5.2155	5.2724	5.3278	5.3816	5.4340	5.4851	5.5349	5.5834	5.6308	5.6771	5.7223	5.7666	5.8098	5.8521	5.8935	5.9341
	2	4.9644	5.0285	5.0905	5.1507	5.2091	5.2659	5.3210	5.3747	5.4269	5.4778	5.5274	5.5758	5.6230	5.6691	5.7141	5.7581	5.8012	5.8433	5.8846	5.9250
	3	4.9601	5.0240	5.0860	5.1460	5.2043	5.2609	5.3159	5.3694	5.4216	5.4722	5.5217	5.5699	5.6170	5.6629	5.7078	5.7517	5.7946	5.8366	5.8777	5.9179
	4	4.9565	5.0203	5.0821	5.1420	5.2002	5.2566	5.3115	5.3649	5.4169	5.4675	5.5168	5.5649	5.6119	5.6577	5.7025	5.7463	5.7890	5.8309	5.8719	5.9120
	5	4.9533	5.0170	5.0787	5.1385	5.1965	5.2529	5.3077	5.3610	5.4128	5.4633	5.5126	5.5606	5.6074	5.6531	5.6978	5.7415	5.7841	5.8259	5.8668	5.9068
	6	4.9504	5.0140	5.0756	5.1353	5.1933	5.2495	5.3042	5.3574	5.4092	5.4596	5.5087	5.5566	5.6034	5.6490	5.6936	5.7371	5.7797	5.8214	5.8621	5.9021
	7	4.9477	5.0112	5.0728	5.1324	5.1902	5.2464	5.3010	5.3541	5.4058	5.4561	5.5052	5.5530	5.5996	5.6452	5.6897	5.7331	5.7756	5.8172	5.8579	5.8977
	8	4.9452	5.0087	5.0701	5.1297	5.1874	5.2435	5.2981	5.3511	5.4027	5.4529	5.5019	5.5496	5.5962	5.6416	5.6860	5.7294	5.7718	5.8133	5.8539	5.8937
	9	4.9429	5.0063	5.0676	5.1271	5.1848	5.2408	5.2953	5.3482	5.3997	5.4499	5.4988	5.5464	5.5929	5.6383	5.6826	5.7259	5.7683	5.8097	5.8502	5.8899
10^{-5}	1	4.9407	5.0040	5.0653	5.1247	5.1823	5.2383	5.2926	5.3455	5.3969	5.4470	5.4958	5.5434	5.5898	5.6352	5.6794	5.7226	5.7649	5.8063	5.8467	5.8863
	2	4.9230	4.9857	5.0464	5.1052	5.1622	5.2170	5.2714	5.3236	5.3745	5.4240	5.4722	5.5192	5.5650	5.6097	5.6534	5.6961	5.7377	5.7785	5.8183	5.8573
	3	4.9094	4.9716	5.0319	5.0902	5.1468	5.2017	5.2550	5.3069	5.3573	5.4063	5.4541	5.5006	5.5460	5.5903	5.6335	5.6757	5.7169	5.7572	5.7966	5.8351
	4	4.8979	4.9598	5.0196	5.0776	5.1338	5.1883	5.2413	5.2927	5.3427	5.3914	5.4388	5.4849	5.5299	5.5738	5.6167	5.6585	5.6993	5.7392	5.7782	5.8164
	5	4.9878	4.9493	5.0089	5.0665	5.1224	5.1766	5.2292	5.2803	5.3299	5.3783	5.4253	5.4711	5.5158	5.5594	5.6018	5.6433	5.6838	5.7234	5.7621	5.7999
	6	4.8787	4.9399	4.9991	5.0565	5.1120	5.1659	5.2182	5.2690	5.3184	5.3664	5.4131	5.4587	5.5030	5.5463	5.5885	5.6296	5.6698	5.7091	5.7475	5.7850
	7	4.8703	4.9312	4.9902	5.0472	5.1025	5.1561	5.2081	5.2586	5.3077	5.3555	5.4020	5.4472	5.4913	5.5342	5.5762	5.6171	5.6570	5.6960	5.7341	5.7713
	8	4.8625	4.9232	4.9818	5.0386	5.0937	5.1470	5.1988	5.2490	5.2979	5.3453	5.3915	5.4365	5.4803	5.5231	5.5647	5.6053	5.6450	5.6837	5.7216	5.7586
	9	4.8551	4.9156	4.9740	5.0306	5.0853	5.1384	5.1900	5.2400	5.2886	5.3358	5.3818	5.4265	5.4701	5.5126	5.5540	5.5943	5.6338	5.6723	5.7099	5.7466
10^{-4}	1	4.8482	4.9084	4.9666	5.0229	5.0775	5.1303	5.1816	5.2314	5.2798	5.3268	5.3725	5.4170	5.4604	5.5026	5.5438	5.5840	5.6231	5.6614	5.6988	5.7353
	2	4.7922	4.8506	4.9069	4.9614	5.0141	5.0651	5.1145	5.1624	5.2089	5.2541	5.2980	5.3406	5.3821	5.4225	5.4619	5.5002	5.5375	5.5739	5.6095	5.6441
	3	4.7493	4.8062	4.8612	4.9142	4.9655	5.0151	5.0631	5.1096	5.1547	5.1985	5.2409	5.2822	5.3223	5.3612	5.3992	5.4361	5.4720	5.5070	5.5411	5.5744
	4	4.7132	4.7689	4.8227	4.8745	4.9246	4.9730	5.0198	5.0652	5.1091	5.1516	5.1929	5.2330	5.2719	5.3097	5.3464	5.3822	5.4169	5.4508	5.4837	5.5158
	5	4.3814	4.7361	4.7888	4.8396	4.8886	4.9360	4.9818	5.0261	5.0689	5.1105	5.1507	5.1898	5.2276	5.2644	5.3001	5.3348	5.3686	5.4014	5.4333	5.4644
	6	4.6527	4.7065	4.7582	4.8081	4.8562	4.9026	4.9475	4.9908	5.0327	5.0733	5.1127	5.1508	5.1877	5.2236	5.2583	5.2921	5.3249	5.3568	5.3879	5.4180
	7	4.6264	4.6793	4.7301	4.7791	4.8264	4.8719	4.9159	4.9584	4.9995	5.0393	5.0777	5.1160	5.1511	5.1861	5.2200	5.2530	5.2850	5.3160	5.3462	5.3755
	8	4.6019	4.6540	4.7040	4.7522	4.7987	4.8435	4.8867	4.9284	4.9687	5.0076	5.0453	5.0818	5.1171	5.1513	5.1845	5.2166	5.2478	5.2781	5.3075	5.3361
	9	4.5789	4.6302	4.6796	4.7270	4.7727	4.8168	4.8592	4.9002	4.9397	4.9780	5.0149	5.0506	5.0852	5.1187	5.1512	5.1826	5.2131	5.2427	5.2714	5.2992
10^{-3}	1	4.5572	4.6078	4.6565	4.7032	4.7482	4.7915	4.8333	4.8736	4.9124	4.9500	4.9862	5.0213	5.0552	5.0880	5.1197	5.1505	5.1803	5.2092	5.2372	5.2644
	2	4.3832	4.4282	4.4713	4.5125	4.5519	4.5898	4.6260	4.6609	4.6943	4.7264	4.7573	4.7870	4.8155	4.8430	4.8695	4.8950	4.9196	4.9433	4.9662	4.9882
	3	4.2514	4.2923	4.3312	4.3684	4.4038	4.4376	4.4699	4.5007	4.5302	4.5584	4.5854	4.6113	4.6360	4.6597	4.6824	4.7042	4.7251	4.7452	4.7644	4.7829
	4	4.1416	4.1792	4.2148	4.2487	4.2808	4.3114	4.3405	4.3682	4.3945	4.4197	4.4436	4.4664	4.4881	4.5089	4.5287	4.5476	4.5356	4.5829	4.5993	4.6150
	5	4.0462	4.0809	4.1137	4.1448	4.1742	4.2021	4.2285	4.2535	4.2773	4.2998	4.3212	4.3415	4.3608	4.3792	4.3966	4.4131	4.4288	4.4438	4.4580	4.4715
	6	3.9609	3.9932	4.0236	4.0523	4.0793	4.1048	4.1290	4.1517	4.1733	4.1936	4.2129	4.2311	4.2483	4.2646	4.2800	4.2946	4.3084	4.3214	4.3338	4.3455
	7	3.8835	3.9136	3.9418	3.9684	3.9934	4.0168	4.0390	4.0598	4.0794	4.0978	4.1152	4.1316	4.1470	4.1616	4.1753	4.1882	4.2004	4.2119	4.2227	4.2329
	8	3.8123	3.8404	3.8668	3.8914	3.9146	3.9362	3.9566	3.9756	3.9935	4.0103	4.0261	4.0409	4.0548	4.0678	4.0801	4.0916	4.1024	4.1125	4.1220	4.1309
	9	3.7463	3.7726	3.7972	3.8202	3.8416	3.8617	3.8805	3.8980	3.9144	3.9297	3.9440	3.9575	3.9700	3.9817	3.9927	4.0029	4.0125	4.0215	4.0299	4.0377
10^{-2}	1	3.6845	3.7093	3.7323	3.7537	3.7737	3.7923	3.8096	3.8258	3.8408	3.8548	3.8679	3.8801	3.8914	3.9020	3.9119	3.9210	3.9296	3.9375	3.9449	3.9518
	2	3.2153	3.2293	3.2419	3.2534	3.2638	3.2731	3.2816	3.2892	3.2961	3.3023	3.3079	3.3130	3.3175	3.3215	3.3252	3.3285	3.3314	3.3340	3.3364	3.3385
	3	2.8928	2.9012	2.9086	2.9151	2.9209	2.9259	2.9303	2.9342	2.9376	2.9405	2.9431	2.9453	2.9473	2.9489	2.9504	2.9517	2.9528	2.9537	2.9545	2.9552
	4	2.6468	2.6520	2.6565	2.6603	2.6636	2.6664	2.6688	2.6708	2.6725	2.6740	2.6752	2.6762	2.6771	2.6778	2.6784	2.6789	2.6793	2.6797	2.6800	2.6802
	5	2.4491	2.4523	2.4551	2.4574	2.4593	2.4609	2.4622	2.4632	2.4641	2.4648	2.4654	2.4659	2.4663	2.4666	2.4669	2.4671	2.4673	2.4674	2.4675	2.4676
	6	2.2846	2.2867	2.2884	2.2898	2.2909	2.2918	2.2926	2.2931	2.2936	2.2940	2.2943	2.2945	2.2947	2.2948	2.2949	2.2950	2.2951	2.2951	2.2952	2.2952
	7	2.1446	2.1460	2.1470	2.1479	2.1486	2.1491	2.1495	2.1498	2.1500	2.1502	2.1504	3.1505	2.1506	2.1506	2.1507	2.1507	2.1508			
	8	2.0233	2.0241	2.0248	2.0253	2.0257	2.0260	2.0263	2.0264	2.0266	2.0267	2.0267	2.0268	2.0268	2.0269						
	9	1.9165	1.9171	1.9175	1.9178	1.9181	1.9183	1.9184	1.9185	1.9186	1.9186	1.9186									
10^{-1}	1	1.8218	1.8219	1.8222	1.8224	1.8226	1.8227	1.8227	1.8228	1.8228	1.8229										
	2	1.2226	1.2226	1.2226																	

Values of $M(u,\beta)$ are equal to $W(u)$ for u greater than $2 \cdot 10^{-1}$ and all values of β

TABLE 1.—*CONTINUED*

u \ β		10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50
	0	5.9964	6.3595	6.6668	6.9333	7.1684	7.3789	7.5692	7.7431	7.9030	8.0511	8.1890	8.3180	8.4392	8.5535	8.6615	8.7641	8.8616	8.9546	9.0435	9.1286	9.2102
	1	5.9739	6.3335	6.6353	6.8973	7.1279	7.3339	7.5197	7.6891	7.8445	7.9881	8.1215	8.2460	8.3627	8.4725	8.5761	8.6741	8.7671	8.8566	8.9400	9.0206	9.0977
	2	5.9645	6.3213	6.6223	6.8823	7.1111	7.3152	7.4992	7.6667	7.8202	7.9620	8.0935	8.2161	8.3309	8.4388	8.5406	8.6367	8.7279	8.8145	8.8971	8.9758	9.0510
	3	5.9573	6.3127	6.6122	6.8706	7.0982	7.3008	7.4834	7.6495	7.8016	7.9419	8.0720	8.1932	8.3066	8.4130	8.5133	8.6081	8.6978	8.7830	8.8641	8.9414	9.0152
	4	5.9513	6.3054	6.6038	6.8612	7.0873	7.2887	7.4701	7.6350	7.7859	7.9250	8.0539	8.1739	8.2861	8.3913	8.4904	8.5839	8.6725	8.7565	8.8364	8.9125	8.9851
10^{-6}	5	5.9460	6.2990	6.5963	6.8527	7.0777	7.2781	7.4584	7.6222	7.7721	7.9101	8.0379	8.1569	8.2680	8.3722	8.4702	8.5627	8.6502	8.7331	8.8119	8.8870	8.9586
	6	5.9413	6.2932	6.5896	6.8450	7.0691	7.2685	7.4478	7.6106	7.7596	7.8966	8.0235	8.1415	8.2516	8.3549	8.4519	8.5435	8.6300	8.7120	8.7899	8.8640	8.9346
	7	5.9367	6.2879	6.5834	6.8379	7.0611	7.2596	7.4381	7.6000	7.7481	7.8842	8.0102	8.1273	8.2366	8.3390	8.4352	8.5258	8.6115	8.6926	8.7696	8.8428	8.9126
	8	5.9326	6.2830	6.5776	6.8313	7.0537	7.2514	7.4290	7.5901	7.7374	7.8727	7.9979	8.1142	8.2226	8.3242	8.4196	8.5094	8.5942	8.6745	8.7507	8.8231	8.8921
	9	5.9287	6.2783	6.5722	6.8251	7.0467	7.2436	7.4205	7.5809	7.7273	7.8619	7.9863	8.1018	8.2095	8.3103	8.4049	8.4940	8.5781	8.6576	8.7330	8.8047	8.8729
	1	5.9251	6.2739	6.5671	6.8193	7.0402	7.2363	7.4125	7.5721	7.7178	7.8517	7.9753	8.0901	8.1971	8.2972	8.3910	8.4794	8.5628	8.6416	8.7163	8.7872	8.8547
	2	5.8955	6.2385	6.5257	6.7720	6.9870	7.1773	7.3476	7.5013	7.6412	7.7692	7.8871	7.9960	8.0973	8.1914	8.2795	8.3631	8.4397	8.5127	8.5817	8.6469	8.7087
	3	5.8729	6.2113	6.4940	6.7358	6.9463	7.1321	7.2979	7.4472	7.5826	7.7061	7.8195	7.9240	8.0208	8.1106	8.1943	8.2725	8.3458	8.4145	8.4792	8.5401	8.5976
	4	5.8538	6.1884	6.4673	6.7053	6.9120	7.0940	7.2561	7.4016	7.5332	7.6531	7.7627	7.8636	7.9566	8.0428	8.1229	8.1975	8.2671	8.3322	8.3933	8.4507	8.5047
10^{-5}	5	5.8369	6.1682	6.4438	6.6785	6.8818	7.0606	7.2193	7.3615	7.4899	7.6065	7.7129	7.8105	7.9003	7.9833	8.0602	8.1316	8.1982	8.2602	8.3182	8.3725	8.4235
	6	5.8217	6.1500	6.4225	6.6542	6.8546	7.0303	7.1861	7.3253	7.4508	7.5644	7.6679	7.7626	7.8496	7.9298	8.0038	8.0725	8.1362	8.1955	8.2508	8.3024	8.3507
	7	5.8078	6.1332	6.4030	6.6319	6.8296	7.0025	7.1556	7.2921	7.4149	7.5259	7.6267	7.7188	7.8032	7.8807	7.9522	8.0183	8.0795	8.1364	8.1892	8.2384	8.2843
	8	5.7948	6.1177	6.3848	6.6112	6.8063	6.9767	7.1272	7.2613	7.3815	7.4901	7.5886	7.6781	7.7601	7.8353	7.9044	7.9682	8.0271	8.0817	8.1323	8.1792	8.2229
	9	5.7826	6.1030	6.3678	6.5917	6.7844	6.9525	7.1007	7.2324	7.3503	7.4565	7.5527	7.6401	7.7198	7.7928	7.8597	7.9214	7.9782	8.0306	8.0791	8.1241	8.1658
	1	5.7710	6.0892	6.3517	6.5734	6.7638	6.9296	7.0756	7.2051	7.3208	7.4249	7.5189	7.6042	7.6818	7.7527	7.8177	7.8773	7.9321	7.9826	8.0292	8.0723	8.1122
	2	5.6780	5.9778	6.2221	6.4257	6.5982	6.7463	6.8747	6.9869	7.0856	7.1729	7.2504	7.3194	7.3811	7.4364	7.4861	7.5307	7.5709	7.6072	7.6399	7.6695	7.6962
	3	5.6069	5.8928	6.1233	6.3133	6.4725	6.6075	6.7231	6.8227	6.9091	6.9844	7.0502	7.1080	7.1588	7.2035	7.2429	7.2777	7.3084	7.3356	7.3597	7.3809	7.3997
	4	5.5471	5.8214	6.0406	6.2194	6.3677	6.4920	6.5972	6.6868	6.7635	6.8294	6.8862	6.9353	6.9778	7.0146	7.0465	7.0742	7.0982	7.1191	7.1371	7.1528	7.1663
10^{-4}	5	5.4947	5.7589	5.9681	6.1374	6.2763	6.3915	6.4880	6.5692	6.6378	6.6961	6.7456	6.7878	6.8237	6.8544	6.8805	6.9029	6.9219	6.9380	6.9516	6.9635	6.9734
	6	5.4474	5.7026	5.9031	6.0638	6.1945	6.3019	6.3908	6.4648	6.5266	6.5784	6.6218	6.6583	6.6890	6.7147	6.7363	6.7545	6.7696	6.7823	6.7929	6.8017	6.8090
	7	5.4041	5.6511	5.8437	5.9967	6.1201	6.2205	6.3027	6.3704	6.4263	6.4726	6.5109	6.5427	6.5690	6.5907	6.6087	6.6235	6.6357	6.6457	6.6539	6.6605	6.6661
	8	5.3639	5.6034	5.7887	5.9348	6.0515	6.1456	6.2219	6.2841	6.3349	6.3763	6.4103	6.4380	6.4607	6.4791	6.4942	6.5063	6.5162	6.5241	6.5305	6.5357	6.5397
	9	5.3263	5.5588	5.7374	5.8771	5.9878	6.0762	6.1472	6.2044	6.2506	6.2879	6.3181	6.3424	6.3620	6.3777	6.3903	6.4004	6.4084	6.4147	6.4197	6.4236	6.4267
	1	5.2908	5.5168	5.6692	5.8230	5.9281	6.0113	6.0775	6.1303	6.1724	6.2061	6.2330	6.2543	6.2713	6.2848	6.2954	6.3037	6.3102	6.3153	6.3192	6.3222	6.3246
	2	5.0095	5.1861	5.3123	5.4037	5.4701	5.5184	5.5534	5.5788	5.5970	5.6101	5.6193	5.6257	5.6302	5.6333	5.6354	5.6368	5.6377	5.6383	5.6387	5.6390	5.6391
	3	4.8006	4.9437	5.0402	5.1056	5.1498	5.1795	5.1993	5.2124	5.2208	5.2263	5.2297	5.2318	5.2331	5.2339	5.2343	5.2346	5.2347	5.2348	5.2349		5.2349
	4	4.6301	4.7481	4.8235	4.8714	4.9017	4.9205	4.9320	4.9390	4.9430	4.9454	4.9467	4.9474	4.9478	4.9480	4.9481						4.9481
10^{-3}	5	4.4844	4.5830	4.6426	4.6783	4.6993	4.7114	4.7183	4.7220	4.7241	4.7251	4.7256	4.7259	4.7260	4.7260							4.7260
	6	4.3566	4.4396	4.4872	4.5140	4.5288	4.5367	4.5409	4.5429	4.5439	4.5444	4.5446	4.5447									4.5447
	7	4.2425	4.3128	4.3510	4.3714	4.3819	4.3871	4.3896	4.3908	4.3913	4.3916	4.3916										4.3916
	8	4.1395	4.1991	4.2300	4.2455	4.2530	4.2565	4.2580	4.2587	4.2589	4.2590	4.2591										4.2591
	9	4.0450	4.096'	4.1212	4.1331	4.1385	4.1408	4.1417	4.1421	4.1422	4.1423											4.1423
	1	3.9582	4.0020	4.0224	4.0316	4.0355	4.0370	4.0376	4.0378													4.0378
	2	3.3403	3.3607	3.3537	3.3545	3.3547																3.3547
	3	2.9558	2.9586	2.9590																		2.9590
	4	2.6804	2.6812	2.6812																		2.6812
10^{-2}	5	2.4677	2.4679																			2.4679
	6	2.2952																				2.2952

Values of $M(u,\beta)$ are equal to $W(u)$ for u greater than $6 \cdot 10^{-2}$ and all values of β

u \ β		50	52	54	56	58	60	62	64	66	68	70	72	74	76	78	80	82	84	86	88	90
	0	9.2102	9.2886	9.3641	9.4368	9.5069	9.5747	9.6403	9.7037	9.7653	9.8249	9.8829	9.9392	9.9940	10.0473	10.0992	10.1498	10.1992	10.2474	10.2944	10.3404	10.3853
	1	9.0977	9.1716	9.2426	9.3108	9.3765	9.4396	9.5008	9.5598	9.6168	9.6720	9.7255	9.7773	9.8276	9.8764	9.9239	9.9700	10.0148	10.0585	10.1011	10.1425	10.1830
	2	9.0510	9.1231	9.1922	9.2585	9.3223	9.3838	9.4430	9.5001	9.5553	9.6086	[illegible]	9.7102	9.7586	9.8056	9.8512	9.8956	9.9385	9.9803	10.0210	10.0606	10.0992
	3	9.0152	9.0859	9.1535	9.2185	9.2809	9.3409	9.3986	9.4543	9.5081	9.5600	[illegible]	[illegible]	9.7058	9.7513	9.7955	9.8384	9.8800	9.9204	9.9597	9.9979	10.0351
	4	8.9851	9.0545	9.1210	9.1847	9.2459	9.3047	9.3613	9.4158	9.4684	9.5191	9.5681	9.6155	9.6613	9.7057	9.7487	9.7904	9.8308	9.8700	9.9081	9.9452	9.9812
10^{-6}	5	8.9586	9.0269	9.0924	9.1550	9.2152	9.2729	9.3285	9.3819	9.4335	9.4831	9.5311	9.5774	9.6222	9.6655	9.7075	9.7481	9.7876	9.8257	9.8628	9.8988	9.9338
	6	8.9346	9.0020	9.0665	9.1282	9.1874	9.2442	9.2988	9.3513	9.4019	9.4507	9.4977	9.5431	9.5869	9.6293	9.6703	9.7101	9.7486	9.7858	9.8220	9.8571	9.8911
	7	8.9125	8.9791	9.0427	9.1036	9.1619	9.2179	9.2716	9.3232	9.3730	9.4208	9.4670	9.5115	9.5545	9.5961	9.6362	9.6751	9.7127	9.7492	9.7845	9.8187	9.8519
	8	8.8921	8.9578	9.0206	9.0807	9.1382	9.1933	9.2463	9.2971	9.3460	9.3931	9.4385	9.4822	9.5244	9.5652	9.6046	9.6426	9.6795	9.7151	9.7497	9.7831	9.8156
	9	8.8729	8.9378	9.0000	9.0592	9.1160	9.1703	9.2225	9.2726	9.3208	9.3671	9.4118	9.4548	9.4962	9.5362	9.5749	9.6122	9.6483	9.6833	9.7171	9.7498	9.7815
	1	8.8547	8.9190	8.9803	9.0389	9.0949	9.1486	9.2001	9.2495	9.2970	9.3426	9.3865	9.4288	9.4696	9.5089	9.5469	9.5835	9.6189	9.6532	9.6863	9.7183	9.7494
	2	8.7087	8.7673	8.8229	8.8759	8.9263	8.9743	9.0202	9.0640	9.1059	9.1461	9.1845	9.2213	9.2566	9.2906	9.3230	9.3542	9.3843	9.4132	9.4410	9.4677	9.4935
	3	8.5976	8.6519	8.7033	8.7521	8.7983	8.8422	8.8839	8.9237	8.9615	8.9975	9.0319	9.0647	9.0960	9.1260	9.1546	9.1819	9.2081	9.2332	9.2572	9.2802	9.3023
	4	8.5047	8.5555	8.6035	8.6488	8.6916	8.7321	8.7705	8.8069	8.8414	8.8742	8.9053	8.9349	8.9630	8.9898	9.0153	9.0396	9.0628	9.0848	9.1059	9.1260	9.1451
10^{-5}	5	8.4235	8.4713	8.5163	8.5587	8.5986	8.6362	8.6717	8.7053	8.7370	8.7671	8.7955	8.8224	8.8479	8.8721	8.8950	8.9167	8.9374	8.9570	8.9756	8.9932	9.0100
	6	8.3507	8.3959	8.4383	8.4780	8.5154	8.5505	8.5836	8.6147	8.6440	8.6716	8.6977	8.7223	8.7455	8.7675	8.7882	8.8078	8.8263	8.8438	8.8603	8.8760	8.8906
	7	8.2843	8.3271	8.3672	8.4046	8.4397	8.4726	8.5034	8.5324	8.5596	8.5851	8.6091	8.6317	8.6530	8.6730	8.6918	8.7095	8.7262	8.7419	8.7567	8.7706	8.7838
	8	8.2229	8.2636	8.3016	8.3370	8.3700	8.4009	8.4297	8.4568	8.4821	8.5057	8.5279	8.5487	8.5682	8.5865	8.6036	8.6197	8.6348	8.6490	8.6623	8.6747	8.6864
	9	8.1658	8.2045	8.2405	8.2740	8.3052	8.3343	8.3614	8.3866	8.4102	8.4322	8.4528	8.4720	8.4899	8.5067	8.5223	8.5370	8.5507	8.5635	8.5754	8.5866	8.5970
	1	8.1122	8.1491	8.1833	8.2151	8.2446	8.2720	8.2974	8.3211	8.3431	8.3636	8.3827	8.4005	8.4170	8.4324	8.4468	8.4601	8.4726	8.4842	8.4949	8.5050	8.5143
	2	7.6962	7.7203	7.7421	7.7618	7.7797	7.7958	7.8104	7.8236	7.8355	7.8463	7.8560	7.8648	7.8727	7.8798	7.8862	7.8920	7.8972	7.9019	7.9061	7.9098	7.9132
	3	7.3997	7.4163	7.4309	7.4439	7.4553	7.4654	7.4742	7.4820	7.4889	7.4949	7.5002	7.5048	7.5088	7.5124	7.5154	7.5181	7.5204	7.5225	7.5242	7.5258	7.5271
	4	7.1663	7.1780	7.1881	7.1968	7.2043	7.2106	7.2163	7.2211	7.2251	7.2286	7.2315	7.2341	7.2362	7.2380	7.2395	7.2408	7.2419	7.2428	7.2436	7.2442	7.2447
10^{-4}	5	6.9734	6.9818	6.9888	6.9948	6.9998	7.0040	7.0075	7.0105	7.0129	7.0150	7.0167	7.0181	7.0192	7.0202	7.0209	7.0216	7.0221	7.0225	7.0228	7.0231	7.0233
	6	6.8090	6.8151	6.8201	6.8242	6.8276	6.8304	6.8327	6.8345	6.8360	6.8372	6.8382	[illegible]	6.8398	6.8401	6.8405	6.8408	6.8411	6.8413	6.8414	6.8416	6.8417
	7	6.6661	6.6705	6.6741	6.6770	6.6793	6.6811	6.6826	6.6838	6.6847	6.6854	6.6860	[illegible]	6.6868	6.6871	6.6873	6.6874	6.6875	6.6876	6.6877	6.6878	6.6878
	8	6.5397	6.5430	6.5456	6.5476	6.5492	6.5504	6.5514	6.5521	6.5527	6.5531	6.5535	6.5537	6.5539	6.5541	6.5542	6.5543	6.5543	6.5544	6.5544	6.5544	6.5544
	9	6.4267	6.4291	6.4310	6.4324	6.4335	6.4343	6.4350	6.4354	6.4358	6.4361	6.4363	6.4364	6.4365	6.4366	6.4366	6.4367	6.4367	6.4367	6.4368		6.4368
10^{-3}	1	6.3246	6.3263	6.3277	6.3287	6.3294	6.3300	6.3304	6.3307	6.3310	6.3311	6.3312	6.3313	6.3314	6.3314	6.3315						6.3315
	2	5.6391	5.6392	5.6393	5.6393	5.6393																5.6393

Values of $M(u,\beta)$ are equal to $W(u)$ for u greater than $2 \cdot 10^{-3}$ and all values of β

infinitely deep, Eq. 2a reduces to

$$s = \frac{Q}{8 \pi K (1 - d)} E\left(u, \frac{1}{r}, \frac{d}{r}, \frac{z}{r}\right) \qquad (7)$$

in which the function E is given by the four M terms of Eq. 2a.

Eq. 7 states, in effect, that in the initial period of pumping, the drawdown around a partially penetrating well would be the same as though the aquifer were infinitely deep. The length of the initial period depends on the penetration depth of the pumped well, the depth of the observation point, and the thickness of the formation, as well as the hydraulic properties of the squifer.

Equation of Drawdown for Relatively Large Values of Time.—It can be shown (see tables of $W(u,y)$) that for $u < ((\pi r/b)^2/20)$, the function $W(u, n\pi r/b)$ can, for all practical purposes, be replaced by $2 K_o (n \pi r/b)$; in which case the series in Eq. 1 becomes independent of time. Hence, for $u < (1/2)(r/b)^2$, that is, t $(b^2 S_s/(2 K))$, Eq. 1 becomes

$$s = \frac{Q}{4 \pi K b} \left[W(u) + f_s\left(\frac{r}{b}, \frac{1}{b}, \frac{d}{b}, \frac{z}{b}\right)\right] \qquad (8a)$$

in which

$$f_s = \frac{4 b}{\pi(1 - d)} \sum_{n=1}^{\infty} \frac{1}{n} K_o\left(\frac{n \pi r}{b}\right)\left(\sin \frac{n \pi l}{b} - \sin \frac{n \pi d}{b}\right) \cos \frac{n \pi z}{b} \qquad (8b)$$

in which K_o is the zero-order modified Bessel function of the second kind.

Eq. 8a shows that in this range of time, the rate of change of drawdown is the same as though the pumped well completely penetrated the aquifer. In other words, the effect of partial penetration on the drawdown has attained its maximum value.

AVERAGE DRAWDOWN IN OBSERVATION WELLS

The water level in an observation well reflects the average drawdown in the aquifer profile that is occupied by the screened portion (or perforated section of the casing) of the well. The average drawdown $\overline{s}$ in an observation well screened between the depths l' and d' (l' > d') can be obtained by integrating the equation of drawdown in piezometers with respect to z between the limits d' and l', and then dividing the result by (l' - d'). If this operation is performed on Eq. 1, the result immediately can be seen to be:

$$\overline{s} = \frac{Q}{4 \pi K b} \left[W(u), + \overline{f}\left(u, \frac{r}{b}, \frac{1}{b}, \frac{d}{b}, \frac{l'}{b}, \frac{d'}{b}\right]\right. \qquad (9a)$$

in which

$$\overline{f} = \frac{2 b^2}{\pi^2 (1 - d)(l' - d)} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\sin \frac{n \pi l}{b} - \sin \frac{n \pi d}{b}\right)$$

$$\left(\sin \frac{n \pi l'}{b} - \sin \frac{n \pi d'}{b}\right) W\left(u, \frac{n \pi r}{b}\right) \qquad (9b)$$

The result of averaging Eq. 2a is rather complicated and will not be given here. Of interest in this result, however, is the part representing the average drawdown for relatively small values of time. This part is given below.

Equation of Average Drawdown for Relatively Small Values of Time.—For $t < \left[(2b - l - l')^2 S_s / (20 K)\right]$, the f' terms of Eq. 2a drop out for all practical purposes. The result of averaging the remaining four M terms is given by

$$\bar{s} = \frac{Q}{8 \pi K(l - d)(l' - d')} \left[F\left(u, \frac{l + l'}{r}, \frac{l - l'}{r}\right) - F\left(u, \frac{d + l'}{r}, \frac{d - l'}{r}\right) + F\left(u, \frac{d + d'}{r}, \frac{d - d'}{r}\right) - F\left(u, \frac{l + d'}{r}, \frac{l - d'}{r}\right)\right] \qquad (10a)$$

in which

$$F(u, \beta, \alpha) = r \left\{\beta M(u, \beta) - \alpha M(u, \alpha) + 2\left[\sqrt{y}\, \mathrm{erfc}(\sqrt{y\, u}) - x\, \mathrm{erfc}(\sqrt{x\, u}) + \frac{e^{-xu} - e^{-yu}}{\sqrt{\pi u}}\right]\right\} \qquad (10b)$$

in which $x = 1 + \beta^2$ and $y = 1 + \alpha^2$, erfc is the complementary error function (1 - erf), and M is the function defined by Eq. 3a.

Eq. 10a also gives the average drawdown in observation wells tapping an infinitely deep aquifer for the whole range of pumping time.

For $d = d' = 0$, Eq. 7 as well as Eq. 10, will reduce to the corresponding special cases obtained by K. F. Saad[11] in his treatment of flow in thick artesian aquifers.

Computations in Eq. 10a show that if $(l'/l) < 2$, results sufficiently accurate for practical application can be obtained by using the following approximate equation

$$\bar{s} \approx \frac{Q}{8 \pi K(l - d)} \bar{E}\left(u, \frac{l}{r}, \frac{d}{r}, \frac{l'}{r}, \frac{d'}{r}\right) \qquad (11)$$

in which $\bar{E}$ is the value of the function E of Eq. 7, in which the value of z is replaced by $(1/2)(l' + d')$. In other words, the average drawdown in an observation well screened between the depths l' and d' can be approximated by the average of drawdowns registered in two piezometers whose penetration depths are l' and d' respectively, provided that $(l'/l) < 2$.

Also, if $(r/l) > 1$ and $(l'/l) < 1$, the average drawdown in the observation well can, for all practical purposes, be taken as that given by Eq. 7, with the value of z arbitrarily chosen between l' and zero. The choice is generally made so as to simplify the equation, which in certain cases may take the following form:

$$\bar{s} \approx c\, M(u, \beta) \qquad (12)$$

in which case the equation is valid for $t < (2b - r\beta)^2 S_s/(20 K)$, c and β being constants that depend on the parameters of the flow system under consideration. For example, if $l = 3d$, a choice of $z = d$ will reduce Eq. 7 to the following

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11 "Nonsteady Flow Toward Wells Which Partially Penetrate Thick Artesian Aquifers," by K. F. Saad, thesis presented to the New Mexico Institute of Mining and Technology, at Socorro, N. M., in 1960, in partial fulfilment of the requirements of the degree of Master of Science.

$$\bar{s} \approx \frac{3\,Q}{16\,\pi\,K\,l}\; M\left(u, \frac{4\,l}{3\,r}\right) \qquad (12a)$$

Also, if d = 0, a choice of z = 0 will result in

$$\bar{s} \approx \frac{Q}{4\,\pi K\,l}\; M\left(u, \frac{l}{r}\right), \qquad (12b)$$

whereas, a choice of z = l will give

$$\bar{s} \approx \frac{Q}{8\,\pi K\,l}\; M\left(u, \frac{2\,l}{r}\right) \qquad (12c)$$

Equation of Average Drawdown for Relatively Large Values of Time.—Because, for large values of time, that is for $t > \left(b^2\, S_s/(2\,K)\right)$, $W\left(u, \frac{n\,\pi r}{b}\right)$ can be approximated very closely by $2\,K_o\left(\frac{n\,\pi r}{b}\right)$, Eq. 9a will, in this range of time, become

$$\bar{s} = \frac{Q}{4\,\pi K\,b} \left\{ W(u) + \bar{f}_s\left(\frac{r}{b}, \frac{l}{b}, \frac{d}{b}, \frac{l'}{b}, \frac{d'}{b}\right)\right\} \qquad (13a)$$

in which

$$\bar{f}_s = \frac{4\,b^2}{\pi^2\,(l - d)(l' - d')} \sum_{n=1}^{\infty} \frac{1}{n^2}\, K_o\left(\frac{n\,\pi\,r}{b}\right)$$
$$\left[\sin\frac{n\,\pi\,l}{b} - \sin\frac{n\,\pi\,d}{b}\right] \left[\sin\frac{n\,\pi\,l'}{b} - \sin\frac{n\,\pi\,d'}{b}\right] \qquad (13b)$$

The observation made in the paragraph following Eq. 8a holds for Eq. 13a also.

DRAWDOWN IN PIEZOMETERS OR WELLS FOR (r/b) > 1.5

For relatively large distances, that is (r/b) > 1.5, the equation of drawdown has been shown[12] to be:

$$s = \bar{s} = \frac{Q}{4\,\pi K\,b}\, W(u) \qquad (14)$$

In fact, Eq. 14 gives results sufficiently accurate for practical purposes even for (r/b) as small as one, provided $u < 0.1\,(r/b)^2$. Eq. 14 is the same as it would be if the pumped well completely penetrated the aquifer (Theis formula).[13] In other words, the actual three-dimensional flow pattern changes to a radial type and hardly distinguished from that of a radial system at a dis-

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12 "Nonsteady Flow to a Well Partially Penetrating an Infinte Leaky Aquifer," by M. S. Hantush, Proceedings, Iraq i Scientific Soc., 1957, p. 10; also reprinted by New Mexico Inst. of Mining and Tech., Socorro, N. Mex.

13 "Groundwater Hydrology," by David K. Todd, John Wiley and Sons, Inc., New York, 1959, p. 90; also "Arid Zone Hydrology, Recent Development," by H. Schoeller, UNESCO, Paris, France, 1959, p. 37.

tance from the pumped well equal to or greater than 1.5 times the thickness of the aquifer.

RECOVERY EQUATIONS

If t and t' are the time, reckoned respectively from the commencement and end of pumping, the residual drawdown s' in a piezometer during recovery can be shown to be

$$s' = s(t) - s(t') \quad \ldots\ldots\ldots\ldots\ldots \quad (15)$$

Similarly, the average residual drawdown $\bar{s}'$ is an observation well is

$$\bar{s}' = \bar{s}(t) - \bar{s}(t') \quad \ldots\ldots\ldots\ldots\ldots \quad (16)$$

in which $t = t_0 + t'$ and t_0 is the time at which the pumping has ceased. Thus, the recovery equation corresponding to any of the drawdown equations discussed previously can be formulated readily, subject to the same time criteria. For example, by using Eq. 15 and Eq. 1, the general equation of recovery in a piezometer of penetration depth equal to z can be written immediately as

$$s' = \frac{Q}{4\pi K b}\left\{W(u) - W(u') + f\left(u, \frac{r}{b}, \frac{l}{b}, \frac{d}{b}, \frac{z}{b}\right) - f\left(u', \frac{r}{b}, \frac{l}{b}, \frac{d}{b}, \frac{z}{b}\right)\right\} \quad \ldots\ldots\ldots\ldots \quad (17)$$

in which u' is the value of u after replacing t by t'.

If Eq. 2a is used instead of Eq. 1 in conjunction with Eq. 15, the general recovery equation can be put in the following alternative form:

$$s' = \frac{Q}{8\pi K(l - d)}\left\{(\text{terms of Eq. 2a}) - (\text{same terms, with } u' \text{ replacing } u)\right\} \quad \ldots\ldots\ldots \quad (18)$$

A third form can be obtained by subtracting Eq. 2a, with u' replacing u, from Eq. 1.

Whereas Eq. 17 is suitable for computation when t' is large, Eq. 18 is suitable for small values of t; that is $(t_0 + t')$. The third form is suitable for computation when t is large and t' is small.

Other equations can be obtained, of course, by using drawdown equations for the different time criteria.

The preceding equations give the recovery in piezometers. The recovery in observation wells can be obtained similarly by using the appropriate average drawdown equation.

CONCLUSIONS

A more general equation than has previously been available for the drawdown around a steady well partially penetrating an artesian aquifer of uniform thickness and infinite in areal extent has been developed. Additional parameters defining the length and space position of the water-entry face (screened

section) of the pumped well, as well as of the observation wells, have been introduced. The drawdown in piezometers and the average drawdown in observation wells has been expressed analytically in forms amenable to relatively easy computation. The effect of partial penetration on the drawdown around a pumping well is shown in Figs. 2 and 3.

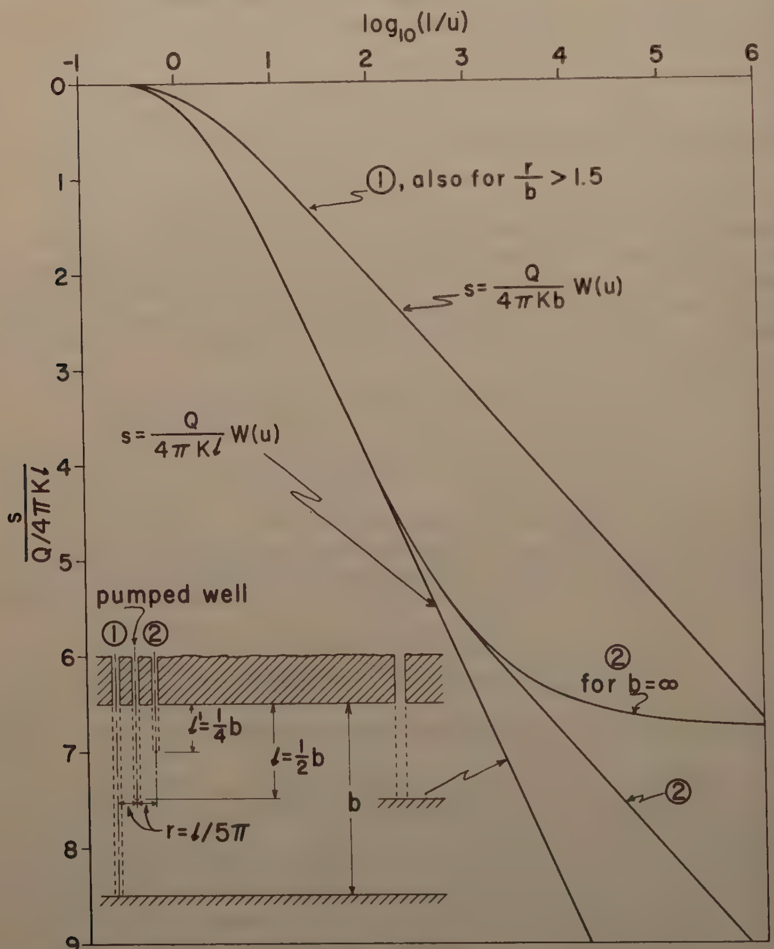

FIG. 2.—TIME DRAWDOWN VARIATION IN PARTIALLY PENETRATING WELLS

The average drawdown developed in an observation well, however close it may be to the partially penetrating pumped well, is given by the Theis formula (Eq. 14), provided that the observation well is screened throughout the aquifer. The same is true in the case of a well located at $(r/b) > 1.5$, regardless of the space position of its screen. In other words, the average drawdow

in such wells is not affected by partial penetration. It is the same as though the pumped well completely penetrated the aquifer (see curve 1 of Fig. 2).

Regardless of the location of the wells and the space position of their screens, the time-drawdown curves will, at relatively large values of time $[t > b^2\ S_s/2\ K)]$, have approximately the same slope. This slope is the same as would obtain if the pumped well completely penetrated the aquifer. In other

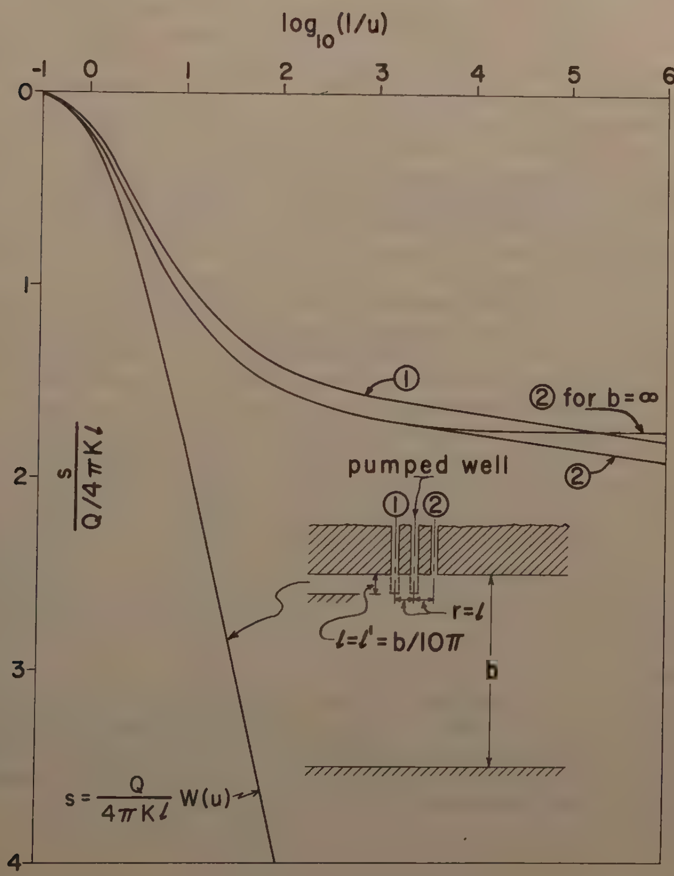

FIG. 3.—TIME-DRAWDOWN VARIATION IN WELLS OF DIFFERENT PENETRATION DEPTHS IN AN ARTESIAN AQUIFER

words, the effect of partial penetration has attained its maximum value (see curves 1 and 2 of Fig. 2).

If the observation well is not relatively distant ($r/b < 1.5$) or is not screened throughout the aquifer, the variation of the average drawdown with the logarithm of time will have the trend shown by the curve labeled "2" in Fig. 2 and curves 1 and 2 of Fig. 3. During the early period of pumping and before the

inflection of the curves appears, such time-drawdown curves have the same general appearance as curves for the case of complete penetration (Theis formula); the Theis formula, of course, is inapplicable. Even in the very early period of pumping, when one may be tempted (for the purpose of applying the Theis formula) to assume that the aquifer ends at the bottom of the pumped well, it cannot be applied except in the case in which the geometry of the flow system is such that the drawdown equation for relatively short time reduces to the form of Eq. 12b, in which case the validity of using the Theis formula is assured only in the range (see Eqs. 6 and 12b) where $t < \left(l^2 S_s/(20\ K)\right)$. However, if the drawdown equation reduces to the form of Eq. 12, the Theis formula, if modified in accordance with Eqs. 12 and 6, can be used to advantage, provided that $t < \left[(r\beta)^2 S_s/(20\ K)\right]$.

Fig. 3 compares the drawdowns observed in two equally distant wells, one of zero penetration and the other screened throughout its depth of penetration. It shows that two wells equally distant from a partially penetrating pumping well may register two different drawdowns. In fact, depending on the length and the relative position of the screens, it is possible for a more distant well to reflect a greater drawdown.

The effects of partial penetration resemble the effects of leakage from storage in a thick, semipervious confining layer.[14] Also, if the curve inflection is apparent, but the period of observation is not long enough to establish the ultimate straight line variation on a semilogarithmic time-drawdown plot, the effects of partial penetration resemble the effects of some kind of recharge boundary, such as induced infiltration from beds of streams or lakes,[15] or recharge from water-bearing strata supplying leakage[16] through semipervious confining beds. The same general effects are observed if the wells completely penetrate a sloping water-table aquifer or an aquifer of nonuniform thickness.[17] Thus, without sufficient information about a flow system that is being studied, observational drawdown trends may be interpreted in several ways. Indiscriminate use of such data may give erroneous and, in many cases, unreasonable results. This, of course, leads to the vexations that arise when attempts are made to force the application of formulas to cases to which they do not apply.

The theory of unsteady flow towards wells partially penetrating an infinite artesian aquifer will be used to outline, in a subsequent paper, methods for the determination of the formation coefficients, as well as the thickness of the water-bearing formation. Applications of these methods will be illustrated by analyzing data from ground-water basins in New Mexico.

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14 "Modification of the Theory of Leaky Aquifers," by M. S. Hantush, Journal of Geophysical Research, Vol. 65, 1960, p. 3713.

15 "Analysis of Data From Pumping Wells Near a River," by M. S. Hantush, Journal of Geophysical Research, Vol. 64, 1959, p. 1921.

16 "Nonsteady Radial Flow in an Infinite Leaky Aquifer," by M. S. Hantush and C. E. Jacob, Transactions, Amer. Geophys. Union, Vol. 36, 1955, p. 95.

17 "Ground-Water Flow in Sands of Nonuniform Thickness," by M. S. Hantush; to be published in the Journal of Geophysical Research, Vol. 66, 1961 or 1962.

APPENDIX - NOTATION

The following is a list of the major symbols used in the text of the paper:

β = a parameter of the function $M(u,\beta)$, also a constant that depends on one or all of the variables, l, l', d, d', z, r, b;

b = thickness of aquifer, dimension L;

c = constant, depending on the well discharge, length of well screen, and on the hydraulic conductivity of the aquifer, dimension L;

d = depth from the top of the aquifer of the unscreened portion (unperforated section of the casing) of the pumped well, L;

d' = depth from the top of the aquifer of the unscreened portion of the observation well, L;

$\mathrm{erf}(x)$ = $\dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-y^2}\,dy$ = the error function;

$\mathrm{erfc}(x)$ = $1 - \mathrm{erf}(x)$ = the complement of the error function;

E = $E(u, l/r, d/r, z/r) = M\left[u, (l+z)/r\right] - M\left[u, (d+z)/r\right] + M\left[u, (l-z)/r\right] - M\left[u, (d-z)/r\right]$

$\bar{E}$ = $\bar{E}(u, l/r, d/r, l'/r, d'/r) = \left[E \text{ with } z = (l' + d')/2\right]$;

f = $f(u, r/b, l/b, d/b, z/b) = \left[2b/\pi(l-d)\right] \displaystyle\sum_{n=1}^{\infty} (1/n)\left[\sin(n\pi l/b) - \sin(n\pi d/b)\right]\cos(n\pi z/b)\, W(u, n\pi r/b)$;

f_s = $\left[f, \text{ with } 2K_0(n\pi r/b) \text{ replacing } W(u, n\pi r/b)\right]$;

$\bar{f}$ = $\bar{f}(u, r/b, l/b, d/b, l'/b, d'/b) = \left[2b^2/\pi^2(l-d)(l'-d')\right] \displaystyle\sum_{n=1}^{\infty} (1/n^2)\left[\sin(n\pi l/b) - \sin(n\pi d/b)\right]\left[\sin(n\pi l'/b) - \sin(n\pi d'/b)\right] W(u, n\pi r/b)$;

$\bar{f}_s$ = $\left[\bar{f}, \text{ with } 2K_0(n\pi r/b) \text{ replacing } W(u, n\pi r/b)\right]$;

f' = $f'(u, b/r, x/r, z/r) = \displaystyle\sum_{n=1}^{\infty} \Big\{ M\left[u, (2nb+x+z)/r\right] - M\left[u, (2nb-x-z)/r\right] + M\left[u, (2nb+x-z)/r\right]\Big\} - M\left[u, (2nb-x+z)/r\right]$;

K = hydraulic conductivity of the aquifer, L T^{-1}.

$K_0(x)$ = the zero-order modified Bessel function of the second kind;

l	= depth of penetration of the pumped well, L;
l'	= depth of penetration of an observation hole, L;
$M(u, \beta)$	$= \int_u^\infty \frac{e^{-y}}{y} \operatorname{erf}(\beta\sqrt{y})\, dy$, tabular values of which are given in Table 1.
Q	= constant well discharge, $L^3\, T^{-1}$;
r	= radial distance measured from center of well, L;
s	= drawdown of piezometric surface at any time and at any point in the aquifer (drawdown in piezometers), L;
$\overline{s}$	= average drawdown in observation holes, L;
s'	= residual drawdown in piezometers, L;
$\overline{s}'$	= residual drawdown in observation holes, L;
S	$= b\, S_s$ = storage coefficient;
S_s	= specific storage (volume of water released from storage by a unit volume of the aquifer under a unit head decline), L^{-1};
t	= time since pumping started, T;
t_o	= period of pumping, T;
t'	= time since pumping stopped, T;
T	= K b = transmissivity of the aquifer, $L^2\, T^{-1}$;
u	$= \left(r^2 S_s/(4\, K\, t)\right)$;
u'	$= \left(r^2\, S_s/(4\, K\, t')\right)$;
W(u)	$= \int_u^\infty \frac{e^{-y}}{y}\, dy$ = well function of (u) for nonleaky aquifers for which tables are available;
W(u,x)	$= \int_u^\infty \frac{dy}{y} \exp\left(-y - x^2/(4\, y)\right)$ = well function of (u and x) for leaky aquifers for which tables are available; and
z	= vertical coordinate measured from the top of the aquifer, positive downward.

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CONTINUOUS PARABOLIC INTERPOLATION

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SYNOPSIS

Some of the basic work load in TVA dealing with hydrologic data processing or hydraulic computation has been shifted to electronic computers. In particular, preparation of rating tables and conversion of gage-height to discharge in a mean daily discharge program are now routinely performed on computers. Both of these require machine interpolation to hold to a minimum the number of data points read into the computer. Conventional interpolation may be done on a straight line connecting any tow adjacent points, or it may be done on a polynomial of higher degree. Straight-line interpolation requires loading many data points into the computer. Polynomials of high degree require more computer time.

Continuous parabolic interpolation was developed as a compromise between simple interpolation with large input requirement and more complex interpolation with longer computation time. It has an advantage of symmetry with respect to forward and backward interpolation across any interval between data points. The fitting complexity is no more difficult than that required for a simple parabola, which, having an even number of intervals, is not symmetrical in forward and backward interpolation.

The greatest advantage, however, is that the continuous parabolic interpolation method produces a smooth curve through the data points. The method was specifically designed to eliminate the abrupt changes in slope of the conventional interpolation polynomials. These polynomials have discontinuous first derivatives at the data points because the numerical expression of the equations is different for each data-point set and, therefore, for each interpolation interval.

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The continuous parabolic interpolation method was developed by the geometrical device of linearly transforming the equation of a simple parabola. The transformation occurs uniformly across the interpolation interval. At the end of one interval the function is the same as at the beginning of the next. Analytically, the function has a continuous first derivative at the data points.

Testing the continuous parabolic method against two known functions gave satisfactory results. An additional comparative test was performed across long segments of a sine wave. The continuous parabolic function gave results almost as good as those using a conventional cubic equation. They were substantially better than those using a simple parabola.

INTRODUCTION

In the Hydraulic Data Branch of TVA, it is frequently necessary to present information in tabular or graphic form. Tables of ratings, of operating characteristics, or of storage volumes, to give some examples, are matters of frequent concern to hydraulic engineers. There is a vast background of such tables. Many of them were computed using trial-and-error smoothing procedures or were derived using judgment in curve drawing to produce data consistent through their entire range. In shifting engineering computations to electronic computer, whether this be computing a full table from skeleton information, or using tabular information in a program, it is necessary to use interpolation methods that will produce results as similar as possible to those using earlier methods. These early methods, however, are not always easily adaptable to computers, or they may not represent the most economical use of computers.

The interpolation problem in hydraulics is illustrated in as simple an application as stream gaging. As part of the data reduction step, it is necessary tc establish a rating table for conversion of gage height to discharge for each measuring station. This rating is a component part of the data for each station. It is necessary so that conversion of gage height to discharge by any of several different users will always give the same answer. These stream rating tables are often computed by "smoothing second differences" of values pickec from a curve drawn by hand through a plot of discharge measurements.

In the method of continuous parabolic interpolation to be presented here the same starting point is assumed as for the earlier "manual" methods. This starting point is the set of skeleton values picked from the hand-drawn curve The reasons for the development of this method and some of the advantages that may be offered by it are illustrated by the data in Table 1.

The skeleton data shown in Table 1 are given no dimensions or units. Th absolute scales, or magnitudes, are immaterial. There is, however, a uniforn interval for the argument, x. In the body of Table 1 are the results for variou methods of interpolation on both sides of the central value of x = 4. The sim plest method of interpolation is the straight line. This corresponds to readin values from chords rather than arcs that connect the skeleton values. On disadvantage of straight line interpolation is the abrupt change in slope to eithe side of a skeleton point. In Table 1, the first difference changes from 0.4 t 0.8 at the value x = 4. Of course, this can be minimized by reducing the inter val of the argument x for those ranges where curvature is severe. Howeve this means using many more points that have to be selected, punched for inpu to the computer, and then read in. Although computers are fast, costs for dat

preparation and input are roughly proportional to the amount of data used. In basic programs, used many times over, input data should be held to a minimum.

TABLE 1.—COMPARATIVE INTERPOLATIONS

			Skeleton Data X: 1, 2, 3, 4, 5, 6, 7, Y: 3, 4, 6, 10, 18, 28, 38.								
	Straight Line		Simple Parabola						Simple Cubic		
			Forward			Backward					
X	Y	$\Delta_1 Y$	Y	$\Delta_1 Y$	$\Delta_2 Y$	Y	$\Delta_1 Y$	$\Delta_2 Y$	Y	$\Delta_1 Y$	$\Delta_2 Y$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
3.6	8.4		8.16			7.92			7.904		
		0.4		0.43			0.46			0.448	
3.7	8.8		8.59		0.02	8.38		0.04	8.352		0.048
		0.4		0.45			0.50			0.496	
3.8	9.2		9.04		0.02	8.88		0.04	8.848		0.052
		0.4		0.47			0.54			0.548	
3.9	9.6		9.51		0.02	9.42		0.04	9.396		0.056
		0.4		0.49			0.58			0.604	
4.0	10.0		10.00		0.13	10.00		0.13	10.000		0.082
		0.8		0.62			0.71			0.686	
4.1	10.8		10.62		0.04	10.71		0.02	10.686		0.036
		0.8		0.66			0.73			0.722	
4.2	11.6		11.28		0.04	11.44		0.02	11.408		0.032
		0.8		0.70			0.75			0.754	
4.3	12.4		11.98		0.04	12.19		0.02	12.162		0.028
		0.8		0.74			0.77			0.782	
4.4	13.2		12.72			12.96			12.944		
			5-Pt. Lagrangian						Continuous Parabolic		
			Forward			Backward					
3.6			8.0544			7.9424			8.016		
				0.4359			0.4512			0.427	
3.7			8.4903		0.0322	8.3936		0.0416	8.443		0.042
				0.4681			0.4928			0.469	
3.8			8.9584		0.0343	8.8864		0.0424	8.912		0.048
				0.5024			0.5352			0.517	
3.9			9.4608		0.0368	9.4216		0.0432	9.429		0.054
				0.5392			0.5784			0.571	
4.0			10.0000		0.0824	10.0000		0.0746	10.000		0.058
				0.6216			0.6530			0.629	
4.1			10.6216		0.0432	10.6530		0.0380	10.629		0.054
				0.6648			0.6910			0.683	
4.2			11.2864		0.0424	11.3440		0.0360	11.312		0.048
				0.7072			0.7270			0.731	
4.3			11.9939		0.0416	12.0710		0.0340	12.043		0.042
				0.7488			0.7610			0.773	
4.4			12.7424			12.8320			12.816		

A slightly more complex method of interpolation is offered by use of a simple parabola. A parabola is the simplest equation allowing interpolation on an

arc. Linear interpolation is based on two adjacent points, parabolic on three. There are, thus, two intervals between points in which interpolation can be accomplished. The analyst must decide whether to use the right hand or left hand interval. In Table 1 under the "Simple Parabola" are shown interpolated results for both forward interpolation, using the right hand interval, and backward interpolation, using the left hand interval. Because different skeleton values are used, different interpolated values result. Adjacent to each set of interpolated values are shown the first and second differences. Both the forward and backward sets exhibit the same abrupt change of slope at the skeleton value x = 4. This discontinuity in the first difference is shown by the extreme value of 0.13 in the second differences.

The next step upward in complexity is interpolation on a cubic equation. A cubic is fitted to four adjacent points. There are three intervals in which interpolations can be made. By using the central interval, interpolation on a cubic is symmetrical with respect to forward and backward interpolation. The second differences in Table 1, however, show that there is still an abrupt change of slope at the skeleton value x = 4. It is not as extreme as for the parabola, but it still occurs. Also shown in Table 1 is interpolation forward and backward using conventional 5-point Lagrangian interpolation coefficients.[2] The second differences still show the discontinuity in the slope at x = 4, though there has again been a reduction in its magnitude.

It is possible, as demonstrated here, to improve interpolation by increasing the degree of the interpolating polynomial. By doing this, the number of skeleton values can probably be reduced with some savings in cost of input data. But this is accomplished at a cost in increased computer time to solve the more complex equations. The actual increase in time may be slight. But even high speed computers require measurable time for computation, and where computer costs are allocated on the basis of user's time minimum, computer time is desirable.

Continuous parabolic interpolation is the result of one attempt to achieve a compromise between precision and cost. This device is easy to understand and is easily programmed. In TVA it has been programmed and is in use on two computers, an IBM-704 and an LGP-30. It has proven extremely useful in converting gage-height to discharge in machine computation of mean daily discharge from time-gage height abstracts. It is also used in generating full tables from skeleton data.

Results using continuous parabolic interpolation are tabulated with results of other methods in Table 1. Continuous parabolic interpolation is accomplished with a 3-point solution equivalent to fitting a simple parabola. It produces smooth interpolation through the skeleton values as shown by the smoothly varying second differences in Table 1. It is symmetrical in forward and backward interpolation.

GEOMETRICAL DESCRIPTION

In Fig. 1, Function 1, is shown a hypothetical set of skeletal values, numbered 1 to 8, and it is desired to develop some reasonable smooth curve through these points. For interpolation between points 2 and 3, the parabola shown by

2 Methods in Numerical Analysis, by Kaj L. Nielsen, The MacMillan Co., New York 1957.

the dashed line through points 1, 2, and 3 could be used. Also, the parabola through points 2, 3, and 4 could be used for the interval between points 3 and 4. The two parabolas intersect at nearly a right angle at point 3, however, and this is undesirable being an extreme case of the abrupt change in slope as discussed using Table 1.

There is a basic problem of lack of continuity of interpolating polynomials when moving from one set of tabular values to the next. Whenever the set advances by dropping one tabular value on the left and picking up a new value on the right, the interpolation polynomial changes its numerical values. Only if the point to be picked up is a forward projection of the polynomial of the previous set will the transition of the curve from set to set be smooth. This is very rarely true and then only by coincidence.

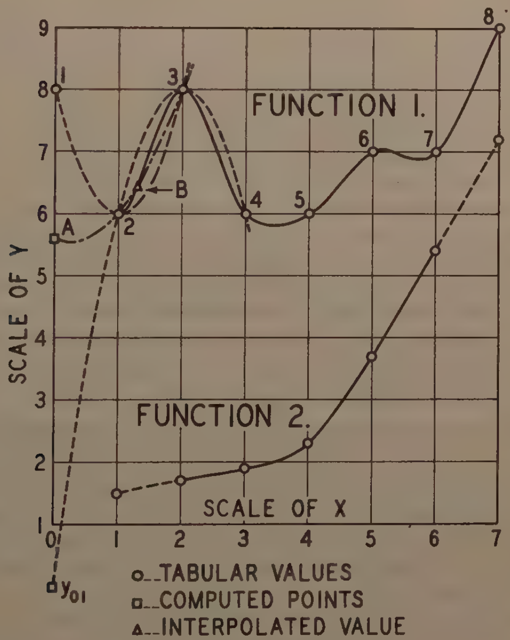

FIG. 1.—CONTINUOUS PARABOLIC INTERPOLATION

The computational device to be presented here solves the problem of continuity through the tabular values by making a linear transformation of the interpolating function as the point of interpolation moves from one value to the next. In Fig. 1, Function 1, the interpolation curve will be continuous if, as interpolation proceeds from point 2 to point 3, a transformation is made from parabola 1-2-3 to parabola 2-3-4. Advancing further, interpolation would leave point 3 on parabola 2-3-4 but be transformed to parabola 3-4-5 by the time point 4 is reached.

The smooth transformation from one parabola to the next can be accomplished very simply. The parabola 2-3-4 was projected back to intersect the ordinate through point 1 at the computed point labeled y_{01}. A parabola through points y_{01}, 2, and 3 is then identical to parabola 2-3-4. If interpolation is made on a parabola that transforms smoothly from parabola 1-2-3 at point 2 to parabola y_{01}-2-3 at point 3, the requirement for continuity is accomplished. The actual interpolation is thus made on a moving parabolic arc that is constantly changing as the point of interpolation changes. This changing parabola always passes through points 2 and 3 while interpolation is between these points. The third point determining the parabola, however, moves along the ordinate through point 1. It ranges from point 1 for interpolation at point 2 to point y_{01} for interpolation at point 3.

In Fig. 1, Function 1, the point A is shown three-tenths of the distance from point 1 to point y_{01} for interpolation at a value $x = 1.3$. Point B is the interpolated value. As the fractional distance from point 2 to point 3 changes with interpolation, the point A moves by the same fractional amount along the ordinate through point 1. The entire interpolating device is moved ahead one set of tabular values, of course, as interpolation passes across point 3. In Fig. 1 the solid curves for Function 1 and Function 2 show the continuous interpolation of values by the above method. In order to compute the values between point 7 and point 8 of Function 1, it was assumed that there was an extension of a straight line through these points. These solid curves through the data points in Fig. 2 are smooth and are reasonable representations of the values between the points. Function 1 was intentionally designed as an extreme test of the interpolation device. Function 2 is an example of a more usual degree of curvature experienced in engineering applications. Here the interpolated curve passes smoothly through a set of tabular values defining a continuously rising curve.

ANALYSIS OF THE INTERPOLATION POLYNOMIAL

The method of interpolation described previously was developed purely along the geometrical lines used in the description. For better definition of the resulting interpolation function, it was analyzed by conventional methods as shown subsequently. As an introduction to this analysis two equations for a parabola, a function of the second degree in x, are needed.

The basic equation for a parabola is the familiar equation

$$y = a + b\,x + c\,x^2 \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

Given any three points, a set of three simultaneous equations can be written expressing the condition that the parabola passes through the points. Solution of the three simultaneous equations give the values for the coefficients a, b, and c.

The equation for a parabola may be written in different ways. One other such way is that resulting from an application of the method of divided differences.[3] In this form the equation is:

$$y = y_n + \left(y_{n+1} - y_n\right)\left(x - x_n\right) + \frac{y_{n+2} - 2\,y_{n+1} + y_n}{2}\left(x - x_n\right)\left(x - x_n - 1\right) \ldots (2)$$

[3] Practical Analysis, by A. Dr. Fr. Willers, Dover Publications, New York, 1948

Eq. 2 is an explicit solution of Eq. 1 by the method of divided differences and represents a parabola passing through the x - y points (x_n, y_n), $(x_n + 1, y_{n+1})$, and $(x_n + 2, y_{n+2})$. Although Eq. 2 is more cumbersome in notation than Eq. 1, they both represent parabolas. Eq. 2 is a better computational form, because a direct solution results by substituting values of the three points through which the parabola passes. A solution of three simultaneous equations is not necessary.

Let the points 1, 2, 3, and 4 of Function 1 in Fig. 1 be designated as having x-values of 0, 1, 2, and 3, and y-values of y_0, y_1, y_2, and y_3. Eq. 2 passing through the points $(1,y_1)$, $(2,y_2)$, and $(3,y_3)$, is found to be

$$y = y_1 + (y_2 - y_1)(x - 1) + \frac{y_3 - 2\,y_2 + y_1}{2}(x-1)(x-2) \quad \ldots\ldots (3)$$

At x = 0, Eq. 3 becomes

$$y_{01} = 3\,y_1 - 3\,y_2 + y_3 \quad \ldots\ldots (4)$$

In Eq. 4, y_{01} is the back-projected point previously described.

The value of the argument, x, for which a value, y, is to be interpolated will always lie between x = 1 and x = 2 for the system as shown. If

$$k = x - 1 \quad \ldots\ldots (5)$$

be the definition of k, then k will vary from 0 to 1 in the interpolation interval. Point A was made to move linearly from point 1 to point y_{01}. This may be written as the more general expression in Eq. 6, using k as defined in Eq. 5:

$$A = y_0 + (y_{01} - y_0)\,k \quad \ldots\ldots (6)$$

The equation of a parabola through the points (0,A), $(1,y_1)$, and $(2,y_2)$ can also be found from Eq. 2. This is the moving parabola on which actual interpolation is made and takes the form in Eq. 7:

$$y = A + (y_1 - A)x + \frac{y_2 - 2\,y_1 + A}{2}(x^2 - x) \quad \ldots\ldots (7)$$

Substituting Eq. 5 for k in Eq. 6 gives

$$A = 2\,y_0 - y_{01} + (y_{01} - y_0)x \quad \ldots\ldots (8)$$

and substituting Eq. 8 for A in Eq. 7 yields the cubic Eq. 9:

$$y = 2\,y_0 - y_{01} - \left(4\,y_0 - \frac{5}{2}\,y_{01} - 2\,y_1 + \frac{1}{2}\,y_2\right)x$$

$$+\left(\frac{5}{2}\,y_0 - 2\,y_{01} - y_1 + \frac{1}{2}\,y_2\right)x^2 - \left(\frac{1}{2}\,y_0 - \frac{1}{2}\,y_{01}\right)x^3 \quad \ldots\ldots (9)$$

At x = 1 Eq. 9 reduces to $y = y_1$, and at x = 2 it reduces to $y - y_2$, thus checking that the cubic interpolating function passes through the two inner-most of a set of four data points.

The first derivative of Eq. 9 is given by Eq. 10:

$$\frac{dy}{dx} = -4\,y_0 + \frac{5}{2}\,y_{01} + 2\,y_1 - \frac{1}{2}\,y_2 + (5\,y_0 - 4\,y_{01} - 2\,y_1 + y_2)x - \left(\frac{3}{2}\,y_0 - \frac{3}{2}\,y_{01}\right)x^2 \quad \ldots (10)$$

and at $x = 1$ this derivative reduces to Eq. 11:

$$\frac{dy}{dx} = -\frac{1}{2} y_0 + \frac{1}{2} y_2 \dots\dots\dots\dots (11)$$

The parabola through $(0,y_0)$, $(1,y_1)$, and $(2,y_2)$ is given in Eq. 12:

$$y = y_0 + (y_1 - y_0)x + \frac{y_2 - 2\,y_1 + y_0}{2}\,(x^2 - x) \dots\dots\dots (12)$$

This parabola also has a first derivative given by Eq. 11 at $x = 1$.

At $x = 2$ Eq. 10 reduces to Eq. 13:

$$\frac{dy}{dx} = \frac{1}{2} y_{01} - 2\,y_1 + \frac{3}{2} y_2 \dots\dots\dots\dots (13)$$

The parabola through $(0,y_{01})$, $(1,y_1)$, and $(2,y_2)$, which is the same as the parabola through $(1,y_1)$, $(2,y_2)$, and $(3,y_3)$, is given in Eq. 14:

$$y = y_{01} + (y_1 - y_{01})\,x + \frac{y_2 - 2\,y_1 + y_{01}}{2}\,(x^2 - x) \dots\dots\dots (14)$$

At $x = 2$ the first derivative of this parabola also has the form given in Eq. 13.

The cubic interpolation function derived here is thus defined. In addition to passing through the two inner-most points of a four-point set it has first derivatives at these points equal to the derivatives of parabolas centered on the points. The continuity of the cubic function, Eq. 9, as it passes from range to range, is the continuity of a parabolic curve through the skeleton value through which the interpolation point is passing.

It should be emphasized at this point that the actual process of interpolation by a continuously transforming parabolic function is accomplished by the sequential solution of the simple Eq. 4, 5, 6 and 7. The final Eq. 7 in this sequence is no more difficult to solve than any other second degree equation. Eq. 9 was developed and analyzed solely to show the complete mathematical form of the interpolating function.

REDUCTION TO COEFFICIENTS

Continuous parabolic interpolation can be reduced to a system of coefficients similar to those of well-known interpolation methods.[4] These coefficients are convenient for intermittent use with a desk computer and for spot checking electronic computer output during initial trials of a new program.

Reduction to coefficients is accomplished by substituting Eq. 4 for y_{01} in Eq. 9. The resultant equation can be arranged by grouping all terms containing y_0, those containing y_1, and so on. The y's can then be factored out, producing the basic Eq. 15:

$$y = C_0\,y_0 + C_1\,y_1 + C_2\,y_2 + C_3\,y_3 \dots\dots\dots\dots (15)$$

The C's in Eq. 15 are defined by the set of Eq. 16:

$$C_0 = 2 - 4\,x + 2.5\,x^2 - 0.5\,x^3 \dots\dots\dots\dots (16a)$$

4 Methods in Numerical Analysis, by Kaj L. Nielsen, The MacMillan Co., New York 1957.

$$C_1 = -3 + 9.5\,x - 7\,x^2 + 1.5\,x^3 \qquad (16b)$$

$$C_2 = 3 - 8\,x + 6.5\,x^2 - 1.5\,x^3 \qquad (16c)$$

and

$$C_3 = -1 + 2.5\,x + 2\,x^2 + 0.5\,x^3 \qquad (16d)$$

By definition x can vary only from 1 to 2. The coefficients, C, can be computed for any chosen value of x in this interval, and these coefficients multiplied by any set of y's as indicated by Eq. 15 will produce an interpolated value of y for the set at the chosen value of x.

TABLE 2.—COEFFICIENTS FOR CONTINUOUS PARABOLIC INTERPOLATION

x	$-C_0$	C_1	
1.00	0	1.0000000	2.00
1.05	0.0225625	0.9939375	1.95
1.10	0.0405000	0.9765000	1.90
1.15	0.0541875	0.9488125	1.85
1.20	0.0640000	0.9120000	1.80
1.25	0.0703125	0.8671875	1.75
1.30	0.0735000	0.8155000	1.70
1.35	0.0739375	0.7580625	1.65
1.40	0.0720000	0.6960000	1.60
1.45	0.0680625	0.6304375	1.55
1.50	0.0625000	0.5625000	1.50
1.55	0.0556875	0.4933125	1.45
1.60	0.0480000	0.4240000	1.40
1.65	0.0398125	0.3556875	1.35
1.70	0.0315000	0.2895000	1.30
1.75	0.0234375	0.2265625	1.25
1.80	0.0160000	0.1680000	1.20
1.85	0.0095625	0.1149375	1.15
1.90	0.0045000	0.0685000	1.10
1.95	0.0011875	0.0298125	1.05
2.00	0	0	1.00
	$-C_3$	C_2	x

The coefficients were computed for each 5% point of the range of x from 1 to 2. These coefficients are given in Table 2. A numerical example of their use is presented in the following.

EXAMPLE

Problem

Interpolate for z = 2.675 given tabular data:

z	y
2.5	40
2.6	60
2.7	90
2.8	130

Solution

Set up the new variables:

x	y
0	40
1	60
1.75	y
2	90
3	130

$$y = (-.0234375 \times 40) + (.2265625 \times 60) + (.8671875 \times 90)$$
$$+ (-.0703125 \times 130) = \underline{\underline{81.5625}}$$

Note that these coefficients have an inverse symmetry and, therefore, forward or backward interpolation in any 4-point set would produce the same interpolated value.

It is sometimes required that an integral, rather than an interpolated value is desired from a set of tabulated values. Insofar as the continuous parabola represents the function of the tabulated values, its integral may be considered an approximate value of the integral of the tabular function. That is, Eq. 15 may be integrated to give the area under the approximating continuous parabola between some limits of x. Because x varies from 1 to 2, it is convenient to set the lower limit at x = 1.

TABLE 3.—COEFFICIENTS FOR CONTINUOUS PARABOLIC INTEGRATION

x	$-C'_0$	C'_1	C'_2	$-C'_3$
1.00	0	0	0	0
1.05	0.00058411	0.04989818	0.00070599	0.00002005
1.10	0.00217917	0.09920417	0.00312917	0.00015417
1.15	0.00456328	0.14737735	0.00768516	0.00049922
1.20	0.00753333	0.19393333	0.01473333	0.00113333
1.25	0.01090495	0.23844401	0.02457682	0.00211588
1.30	0.01451250	0.28053750	0.03746250	0.00348750
1.35	0.01820912	0.31989818	0.05358099	0.00527005
1.40	0.02186667	0.35626667	0.07306667	0.00746667
1.45	0.02537578	0.38943984	0.09599765	0.01006172
1.50	0.02864584	0.41927083	0.12239583	0.01302083
1.55	0.03160495	0.44566901	0.15222682	0.01629088
1.60	0.03420000	0.46860000	0.18540000	0.01980000
1.65	0.03639662	0.48808568	0.22176849	0.02345755
1.70	0.03817917	0.50420417	0.26112917	0.02715417
1.75	0.03955078	0.51708984	0.30322265	0.03076172
1.80	0.04053333	0.52693333	0.34773333	0.03413333
1.85	0.04116745	0.53398151	0.39428932	0.03710338
1.90	0.04151250	0.53853750	0.44246250	0.03948750
1.95	0.04164662	0.54096068	0.49176849	0.04108255
2.00	0.04166667	0.54166667	0.54166667	0.04166667

For this condition the integral is shown in Eq. 17:

$$A = \int_1^x y\, dx = y_0 \int_1^x C_0\, dx + y_1 \int_1^x C_1\, dx + y_2 \int_1^x C_2\, dx + y_3 \int_1^x C_3\, dx \quad (17)$$

The integrals in Eq. 17 are easily computed from the definition of the C's in the set of Eqs. 16. The integrals were each evaluated between the lower limit x = 1 and each of the 5% points of the range from x = 1 to x = 2. These values are given in Table 3. A numerical example is presented as follows.

EXAMPLE

Problem

Compute the approximate area between z = 2.6 and z = 2.675 for the tabulated function in the example in Table 1.

Solution

Use integration coefficients for $x = 1.75$.

$$A_x = (-.03955078 \times 40) + (.51708984 \times 60) + (.30322265 \times 90) + (-.03076172 \times 130) = 52.734^{+}$$

$$A_z = \frac{z_1 - z_0}{x_1 - x_0} A_x = \frac{0.1}{1} A_x = 5.2734^{+}$$

Multiplication of any set of tabular values of y by a chosen set of coefficients from Table 3 preduces the area under the continuous parabola between the ordinates through x_1 and the x of the chosen set.

It is interesting to note that at the mid-point of an interval, at which $x = 1.5$, the interpolation coefficients reduce to the easily remembered form in Eq. 18.

$$y = \frac{9}{16}(y_1 + y_2) - \frac{1}{16}(y_0 + y_3) \quad \ldots\ldots\ldots\ldots (18)$$

Also, integration for the entire range, from $x = 1$ to $x = 2$, reduces to the easily remembered form in Eq. 19.

$$A = \frac{13}{24}(y_1 + y_2) - \frac{1}{24}(y_0 + y_3) \quad \ldots\ldots\ldots\ldots (19)$$

TESTING THE INTERPOLATING FUNCTION

A practical test of the method of continuous parabolic interpolation is difficult to design. As demonstrated by the data in Table 1, different methods of interpolation give different results. Comparative interpolations cannot be judged against some true function, because in practical application the true function is not known. If it were, there would be little point in an interpolation function, because any desired values could be computed on the true function. Also, there is little point in comparing results of continuous parabolic interpolation against values from hand-drawn curves. There is no assurance that the hand-drawn curve is any nearer to a "true" curve than that obtained by any other method.

It is possible to test the method of continuous parabolic interpolation against known function. Such a test is shown in Table 4. In the first test a trigonometric function was used, with the interval $x = 1$ to $x = 2$ corresponding to 20° of angle. For the points tested, the largest error for interpolation was two-tenths of one percent. The largest error for integration was one-tenth of 1%, with integration over the entire area having an error of three-hundredths of 1%.

The second test in Table 3 was made with an exponential function. Results for this monotonic function, although not as good as for the oscillatory sine function, are still excellent for practical work. The greatest error for the quarter-points of the interval was three-tenths of 1% for interpolation and one-tenth of 1% for integration.

One additional test of interpolation by the continuous parabolic method is shown in Table 5. A known function was again used as a base. Values of the sine of an angle were tabulated for 45°, 90°, 135°, 180°, and 225°. These were used as skeleton data, and the value of the sine was interpolated for each 5th of the interval 90° to 135°, and the interval 135° to 180°.

TABLE 4.—INTERPOLATION AND INTEGRATION CHECK AGAINST KNOWN FUNCTIONS

Function: $y = \sin \frac{x}{a}$ $\qquad A = \int_1^x \sin \frac{x}{a}\, dx = -a \cos \frac{x}{a} + K$

Points of the function used for base data ($a = 9/\pi$ to convert x/a to radians)

Degrees	x	y (Actual)	y (Interpolated)	Percentage Error	A_1^x (Actual)	A_1^x (Interpolated)	Percentage Error
0	0	0					
20	1	0.34202			0.497469	0.497332	0.03
40	2	0.64279					
60	3	0.86603					
Intermediate points for testing							
24	1.2	0.40674	0.40605	0.2	0.074885	0.074818	0.09
30	1.5	0.50000	0.49983	0.03	0.211020	0.210797	0.1
36	1.8	0.58779	0.58826	0.08	0.374341	0.374181	0.04

Function: $y = e^{ax}$ $\qquad A = \int_1^x e^{ax}\, dx = \frac{1}{a} e^{ax} + K$

Points of the function used for base data (Assume a = 0.5)

Degrees	x	y (Actual)	y (Interpolated)	Percentage Error	A_1^x (Actual)	A_1^x (Interpolated)	Percentage Error
0	0	1.0000					
1	.5	1.6487			2.1392	2.1371	0.1
2	1.0	2.7183					
3	1.5	4.4817					
Intermediate points for testing							
1.25	.625	1.8682	1.8702	0.1	0.4390	0.4395	0.1
1.5	.75	2.1170	2.1138	0.2	0.9366	0.9370	0.04
1.75	.875	2.3989	2.3923	0.3	1.5004	1.4994	0.07

TABLE 5.—TEST BY COMPARATIVE INTERPOLATIONS AGAINST KNOWN FUNCTION

Skeleton points: Degrees: 45, 90, 135, 180, 225

Sine: .70711, 1.000, .70711, 0.000, -.70711

Angle, in degrees (1)	Sine (2)	X (3)	St. Line		Simple Parabola		Continuous Parabola		Cubic	
			Ordinate (4)	Error (5)	Ordinate (6)	Error (7)	Ordinate (8)	Error (9)	Ordinate (10)	Error (11)
99	0.98769	1.2	0.94142	0.04627	0.98828	-0.00059	0.98554	0.00215	0.98280	0.0048
108	0.95106	1.4	0.88284	0.06822	0.95314	-0.00208	0.94490	0.00616	0.94353	0.0075
117	0.89101	1.6	0.82427	0.06674	0.89456	-0.00355	0.88221	0.00880	0.88358	0.0074
126	0.80902	1.8	0.76569	0.04333	0.81255	-0.00353	0.80157	0.00745	0.80432	0.0047
135	0.70711	...	...	...	...	...	...	...	...	...
144	0.58779	1.2	0.56569	0.02210	0.59883	-0.01104	0.59220	-0.00441	0.58557	0.0022
153	0.45399	1.4	0.42427	0.02972	0.47397	-0.01998	0.45409	-0.00010	0.45077	0.0032
162	0.30902	1.6	0.28284	0.02618	0.33255	-0.02353	0.30273	0.00629	0.30604	0.0029
171	0.15643	1.8	0.14142	0.01501	0.17456	-0.01813	0.14805	0.00838	0.15468	0.0017
Sum of Magnitudes				0.31757		0.08243		0.04374		0.034

Values were interpolated by using a straight line across the interval, a simple parabola, a continuous parabola, and a cubic equation. The results for the different methods are shown in Table 5. In addition to the interpolated value, the errors from the true sine values are shown and the sum of the magnitudes of the errors.

Straight line interpolation produces large errors across such long arcs of a function. The sum of the error magnitudes was 0.31757. A simple parabola has smaller errors, the sum of the magnitude being 0.08243. A continuous parabola has somewhat smaller errors. The sum of the magnitudes was roughly half that for the simple parabola, being 0.04374. The largest individual error was 0.00880 as against 0.02353 for a simple parabola. Additional reduction in errors can be accomplished by going to more complex functions. Interpolation using a cubic equation produced an additional reduction to an error sum of 0.03472. Reduction in individually highest errors was from 0.00880 for the continuous parabola to 0.00753 for the cubic. The reduction in error gained by use of a cubic equation in place of a simple parabola was, thus, substantially less than that gained by using a continuous parabola in place of a simple parabola.

CONCLUSIONS

Interpolation can be accomplished in a simple and efficient manner on a continuously transforming parabola, or quadratic equation. Although it can be demonstrated that this is actually a special third degree equation, the fitting is no more difficult than for a simple parabola.

The continuous parabola is recommended as an electronic computer technique. It is a compromise between simple linear interpolation and higher degree interpolation polynomials. This method has the advantage of symmetry with respect to forward and backward interpolation. It is a method specifically designed to produce a smooth curve through the data points. This curve resembles "manually" drawn curves that are also drawn smoothly through the data points. There are no abrupt changes of slope such as result with more conventional interpolation methods.

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USE OF COMPUTERS FOR KANSAS RIVER FLOOD STUDIES

By William L. Northrop,[1] F. ASCE and Clarence W. Timberman[2]

SYNOPSIS

A general digital computer program is described that produces flow hydrographs for an extensive network of basin subareas using unit hydrographs and routing coefficients. This concept is expanded to determination of releases for a system of reservoirs. Development of an electronic analog for reservoir operation is also described.

INTRODUCTION

The section of this paper entitled, "Flow-Determination Program," pertains to the basic computer program for development of hydrographs for a complex network of subareas as used in the analysis of floods in the Kansas River basin. The section entitled "System Reservoir Regulation Program," will describe the problems utilized in the operation of a system of reservoirs, technique of expanding and adapting the computer program in analyzing reservoir operation, and the description of an alternative means such as an electronic flood model of the analog type.

Originally, it was proposed to cover only the use of a computer in developing flood hydrographs in this paper. However, after that program was developed and the concept of its use broadened, it was realized that it could be the foundation for a much broader program for the operation of a system of flood-control reservoirs. Although the program for a reservoir-system operation is in the formative stage and only a first-attempt "workup" has been com-

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pleted, it seemed desirable to present the results of the "workup" to stimulate interest and disseminate ideas towards what appears to be an interesting application of digital computers in the field of hydrology.

Before proceeding with the description of the flow-determination program developed by the Corps of Engineers in the Missouri River Division, brief mention is made of a similar program prepared by David M. Rockwood[3] M. ASCE, in connection with streamflow routing in the Columbia River basin.

Both the Missouri River Division's Flow Determination Program and the North Pacific Division's computer program use the IBM 650 computer to compute flows at numerous points in a large, complex river-basin system. The North Pacific Division's program has certain features that make it especially applicable for forecasting use on a day-to-day basis in which forecasted flows are adjusted periodically to conform to data received from gage reports. Also, the program provides for patterning snowmelt runoff to 6-hr periods from daily values which is particularly applicable in that area. Each program uses different techniques to accomplish basin and channel routings. The distinguishing characteristic of the North Pacific Division's program, which it is desired to emphasize, is that a pilot routine is part of the stored program in the machine memory, whereas the Missouri River Division's program depends on controls in the input-data deck so that the instruction deck is the same regardless of the river-basin system being programmed.

PART I - FLOW-DETERMINATION PROGRAM

Basic Problem Description.—The basic problem programmed is a commor one. For many years engineers concerned with flood-control projects have performed multitudinous and laborious manual computations to determine flow hydrographs on which to base design criteria, analyze operations, anc prepare economic studies. The problem may be stated as determing a hydrograph from contributions of rainfall excess, RE, (rainfall minus losses) from storms or snowmelt over a drainage area, and the modification of the hydrograph as the flow progresses downstream, which includes its combinatio with tributary flows. Because many articles are available on the subject o hydrograph determination and are well-known to engineers practicing in thi field, no attempt will be made to give credit for development of the basi procedures. Revolutionary ideas have not been incorporated in this program The merit of this program lies in the ability of the digital computer to perform the computations quickly and accurately, permitting the undertaking o studies that otherwise would be impracticable because of the sheer amount o labor and time required. The program also permits a superior study becaus of a greater division of subareas of a basin and the investigation of mor cases. Used in streamflow forecasting, results can be produced in time to b of use in determining flood-control operations or for issue of flood warnings

Essential unit hydrographs and flood-routing methods are shown in Corp of Engineer Hydrology Manuals and are well described in R. K. Linsley, F ASCE, M. A. Kohler, F. ASCE, and J. L. H. Paulus,[4] and by B. L. Gilcrest,

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3 "Columbia Basin Streamflow Routing by Computer," by David M. Rockwood, Proceedings, ASCE, Vol. 84, No. WW 5, December, 1958.

4 Applied Hydrology, by R. K. Linsley, M. A. Kohler, and J. L. H. Paulus, McGraw Hill Book Co., Inc., New York, July, 1949.

5 "Flood Routing," by B. L. Gilcrest, Engineering Hydraulics, by Hunter Rouse, Joh Wiley and Sons, New York, Aug. 1950.

as well as other publications. This paper describes the program as written for an IBM 650 computer. The same principles would be applicable to any other medium-sized computer with suitable modifications.

General Features of the Flow-Determination Program.—This program will handle the computation of streamflow hydrographs at numerous control points in a complex network of subareas of a basin in one pass of the input-data deck that contains the base flow hydrographs, subarea unit hydrographs, subarea rainfall excess amounts, and routing coefficients for the reaches under study. Except for the restriction of the number of temporary hydrograph storage locations available, the complexity of the basin system is unlimited as all basin data are stored on cards in the input-data deck and no more computer memory locations are required for a large basin system than for a small one. The time of processing, of course, will increase with the size of the river system and the number of time periods.

To obtain compactness of the input-data deck, a split-word format is used in which a 10-digit-computer word is split into two 5-digit fields. Scaling is determined from a decimal shift specified in the control word on the card. Internally, in machine memory, each field is expanded to 10 digits. Thus, depending on the shift specified, flows can be expressed as follows:

Shift = 0, from 1 to 89,999 cfs in even cfs.
Shift = 1, from 10 to 899,990 cfs in tens of cfs.
Shift = 2, from 100 to 8,999,900 cfs in hundreds of cfs.
Shift = 3, from 1,000 to 89,999,000 cfs in thousands of cfs.

A shift of up to 5 could be used, but ordinarily a shift greater than 2 will not be required. In the Kansas River basin, a shift of 2 for studies of flood flows is convenient, as flows will not exceed upper limit and an accuracy to the nearest 100 cfs is greater than that of the basic-flow data on the larger streams for flows of flood magnitude.

Because a "9" as the first digit of the 5-digit group is used to indicate a negative number, negative numbers and values starting with "9" can be expressed to only 4 significant figures rather than 5. Values in machine memory are always in even second-feet. Because of the decimal-routing coefficients and decimal-rainfall amount, figures other than zero may occur in the nonsignificant low-order digits but will be meaningless as far as accuracy is concerned. As values in memory words are shifted before being punched out as 5-digit fields, the nonsignificant digits will ordinarily be dropped at that time.

The program provides for hydrograph lengths up to 399 time periods utilizing one temporary storage location for hydrographs. As an alternate, up to 199 time periods with 2 temporary storage locations can be used. A card that defines hydrograph length for a particular case is placed at the start of the input-data deck.

A block of an equal number of locations (399) are provided for the unit hydrograph. This is more than needed, but by providing the same length it can be used as working locations to store the hydrograph. This enables a successive-average (Tatum) or lag-average routing to be accomplished using the unit hydrograph times the rainfall-excess computation procedure.

The unit hydrograph times the rainfall-excess subroutine provides for adding the runoff contribution from a subarea accumulatively into the stored hydrograph as computed by multiplying unit hydrograph ordinates by the

rainfall-excess amounts. This is done in turn for each rainfall-excess amount controlled as to timing of the contribution by the time period identification of the rainfall excess. Each amount has its own time identification and does not have to be computed in chronological order. Therefore, it is possible to have several unit hydrographs for a subarea, each to apply to a certain range of rainfall-excess values which provides a very useful procedure in the program. If the rainfall excesses are sorted by ranges and placed in the input-data deck following the appropriate unit hydrograph, then there is, in effect, a unit hydrograph variable in steps as a function of rainfall-excess magnitude, as many natural basins are known to exhibit this nonlinear effect.

Two basic routing procedures are provided in the program. The well-known Muskingum routing with variable coefficients and a level-pool-reservoir routing. The three-coefficient variation of the Muskingum routing equation is used in which:

$$\overline{O}_2 = C_1 I_2 + C_2 I_1 + C_3 \overline{O} \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

in which

$$C_1 = \frac{\Delta t - 2 K X}{2 K (1 - X) + \Delta t} \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$C_2 = \frac{\Delta t + 2 K X}{2 K (1 - X) + \Delta t} \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

$$C_3 = \frac{2 K (1 - X) - \Delta t}{2 K (1 - X) + \Delta t} \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

and

$$C_1 + C_2 + C_3 = 1 \quad \ldots\ldots\ldots\ldots\ldots \quad (5)$$

Coefficients C_1 and C_2 are determined as functions of outflow and C_3 is computed as unity - $(C_1 + C_2)$ The program does not compute the coefficients from Eqs. 2, 3, and 4, which are shown only to indicate their nature but obtains them from the routing table that is part of the input-data deck and is illustrated subsequently in Fig. 4. Twelve points producing 11 straight-line segments can be used to approximate the curves of C_1 and C_2 versus outflow. The program uses the outflow, $\overline{O}_1$ at the beginning of the step as the argument in a table-look-up and linear-interpolation procedure to determine the routing coefficients, then computes a first approximation outflow, $\overline{O}_2$, at the end of the ste in accordance with the basic Eq. 1, in which I_1 and I_2 are the initial and end-of-step inflow, respectively. The initial and first approximation end-of-step outflow is then averaged and used to recompute the final end-of-step outflow.

The determination of the Muskingum routing coefficients to make up the routing tables for the input-data deck will not be covered herein as that procedure is described elsewhere.[4,5] Usually, in practice, the coefficients are obtained from trial routings and flow reconstitutions rather than from Eqs 2, 3, and 4. The references describe the nature and effect of the parameters Δt, time-period length; K, essentially travel time, but also the slope of the storage-outflow curve; and X, which is a weighting factor for the relationship of inflow and outflow of the routing reach to storage in the reach, sometimes called the wedge-storage factor.

For the level-pool-reservoir routing, a routing curve is set up in tabular form to be used by a table-look-up and linear-interpolation procedure. The basic equation is:

$$(2\,V + \overline{O})_2 = (2\,V + \overline{O})_1 + I_1 + I_2 - 2\,\overline{O}_1 \qquad (6)$$

in which V equals storage capacity at some particular elevation in units of 1 sec-ft flowing for the duration of the time period used. Corresponding flow at the same elevation is $\overline{O}$. After the $(2\,V + \overline{O}_1)$ value is obtained computation proceeds step by step to determine successive outflows that replace values in the stored hydrograph locations.

By placing a "punch-out" control card in the input-data deck, the stored hydrograph in machine memory as it exists at that network point, determined by the order of the input-data-deck cards, will be punched out as cards in the output deck. Proper time period indentification will be produced automatically by the program, but scaling and proper identification must be specified on the "punch-out" control card and will be automatically transferred to the output-deck cards.

The program will compute hydrograph volumes (sum of hydrograph ordinates) either for the complete stored hydrograph or subtotals between designated time periods.

In some cases where output data is used as input, it is necessary to shift the time period identification. The program will do this by use of a control card placed in front of the hydrograph data cards.

Generally, negative numbers are not needed for hydrographs, but they may be convenient in some cases to express flow losses or evaporation. The program will make a number negative during the split-word-input routine if the left digit of the 5-digit group is a "9." Thus, negative numbers and positive numbers starting with a "9" are limited to 4 significant figures. The magnitude of the numbers is not limited due to the flexible scaling specified by "Y" in the control word.

Progression of the computations depends on the arrangement of the input-data-deck cards, including placement of necessary control cards, to conform to the schematic diagram of the basin. In most cases, the necessary control is on the same card as the data being in columns 8-10 of the first or control word on the card. In the case of some subroutines where more than one card is required for the data, as the Muskingum or level-pool routings, the cards must be in a certain sequence as only the last card directs the program to the proper subroutine. In the case of the transfer, punch-out, hydrograph length, and "read-in" shift cards, no data are required and only the first word of the card is used, the remaining words being left blank.

Program Features Related to the Computer.—The program instructions, "read-in" from a deck of cards called the instruction deck, are stored in computer memory. The instruction deck directs operation of the computer on data for a particular problem, its data being "read-in" from cards called the data deck. After the instruction deck is once "read-in," any number of batches of data may be processed, even for completely separate basins.

Input data for a particular basin system consists of an actual flow hydrograph at the starting point, if at a midpoint in the basin, subarea unit hydrographs, rainfall-excess amounts, flood-routing coefficients, and various control cards necessary to control the computations from the data and combine

flows in proper order. The assembly of the data deck in proper sequence is, in reality, a part of the programming and must be determined for the particular drainage area under study. Once set up, only manual or machine sorting of the parts of the data subject to change is necessary to run a new problem.

Table 1 shows memory locations in the computer for this program. Various blocks of consecutive memory cells have been reserved for storing hydrograph and unit hydrograph values, rainfall-excess amounts, flood-routing coefficient tables, and so forth. Into the hydrograph locations (abbreviated "H") will initially be read starting hydrograph or base flow information. Results of the computations are always hydrographs at a predetermined network point in the basin. These results will always replace values that are in the "H" locations. The nature of the machine memory is such that as new values are stored they replace values previously stored in the memory cell. The original information, in this case, has been processed and become a part of the desired results. Results at remaining network points are computed in turn and "H" values are punched into cards called the output-data deck before proceeding to the next point in the basin. Usually, many intermediate results are not punched out. They are required in the computation because reach

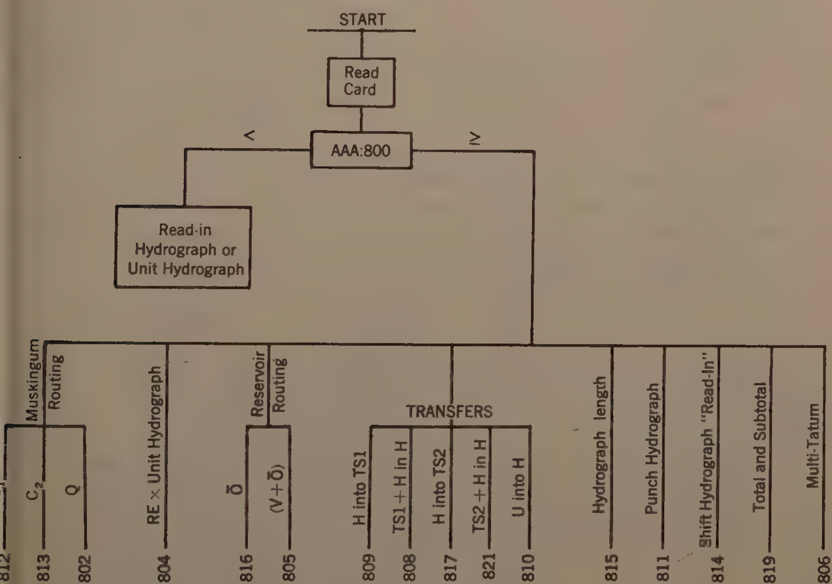
TABLE 1.—MEMORY LOCATIONS

Item (1)	Location (2)
Hydrograph locations	0401 - 0799[a]
Unit hydrograph locations	0801 - 1199[a]
Temporary hydrograph storage	1391 - 1789
RE amounts and time	1879 - 1892
Tables	1096 - 1173
Subroutine starts	1201 - 1250[a]

[a] 400 added to card value by program.

lengths, suitable for computation, are not the same as those determined by gaging stations and/or control points in the river system. Assume there is stored in the "H" locations a hydrograph at a given point in the system and that it is desired to add a flow contribution from a tributary subarea computed from rainfall excess and route the resulting hydrograph to the next downstream network point. The proper unit hydrograph must be the next item in the input-data deck. It will be read into the unit hydrograph ("U") locations, one card at a time, until completely stored. Then the rainfall-excess amounts and their time period identification ("RE" cards) must be on the next cards in sequence. These values are stored, one card at a time, in the "RE" locations. After each "RE" card is "read-in," the machine performs the necessary computation of multiplying unit hydrograph ordinates by "RE" and adding the results accumulatively into values at proper time periods already in the "H" locations. As many cards as necessary may be used following a unit hydrograph until all "RE" amounts have been processed. Note that to route the hydrograph, there must follow in the input-data deck routing-coefficient data for the reach under study that will be read into the tables location. The machine will then perform the routing and the modified hydrograph placed in the "H" location.

Note that temporary storage hydrograph locations are provided called "TS1" or "TS2." Sometimes it is necessary to compute and route tributary hydrographs before combining them with the main stem hydrograph at a given point. In this case, the main stem hydrograph must be set aside in the temporary location provided, then recombined after the tributary computations have been made. Sometimes tributaries to the tributaries may be included. This will require use of the "TS2" locations. Very complicated basin networks requiring more "TS" locations can be processed by punching out the hydrograph on cards and splitting the input-data deck at the point at which this hydrograph will have to be reinserted. The hydrograph on cards will then be



ter completion of any of the above subroutines, program branches back to start and reads the next card in the input-data deck.

FIG. 1.—GENERAL FLOW DIAGRAM

placed in front of the second part of the input-data deck and will be "read-in" at the time this point is reached into the "H" location.

Fig. 1 shows a broad, overall general flow diagram and how control information in the input-data deck directs the operations of the stored program instructions to the proper subroutines.

Fig. 2 shows a detailed flow diagram for one particular subroutine, the "RE" times unit hydrograph computation, that has its first instruction stored in memory location 0804 (Fig. 1). This shows how indexing counters are set up and advanced in looptype computations so that products of "RE" and unit hydrograph ordinates are added into the hydrograph value in "H" for the proper time period.

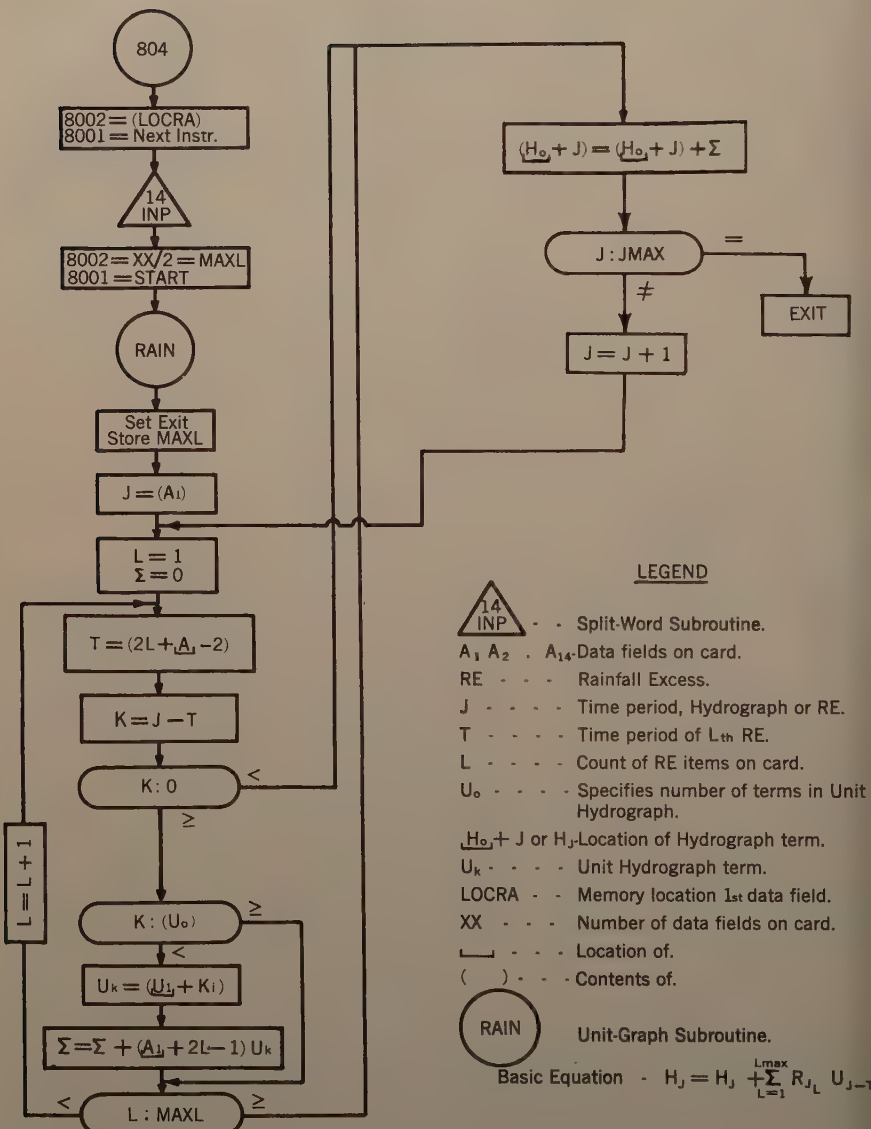


FIG. 2.—FLOW DIAGRAM OF UNIT HYDROGRAPH COMPUTATION

Date Preparation.—Figs. 3 and 4 illustrate format of input data on IBM punch cards. The card illustration is for the first card of a hydrograph "read-in" and is the same as the first hydrograph card given in the hypothetical problem (Appendix A). Tabular data shows setup for additional cards. Unit hydrograph "read-in" is similar to hydrograph "read-in" except as described under "AAA," following. Also shown on Fig. 3, in tabular form, is the setup for rainfall-excess amounts and their time-period identification. Fig. 4 shows, in tabular form, the format for the routing coefficient tables. Note division of computer words into data fields. The first or control word on the card contains 4 fields. These are shifted out and stored in individual words in the computer memory by the program. The function of each of these fields is as follows:

XX Specifies number of data fields in words 2 to 8, inclusive, on the remainder of the card. This information is used by the program to determine when all fields on the card have been processed for branching to the next operation. It also permits unused words at the end of the card to be left blank.

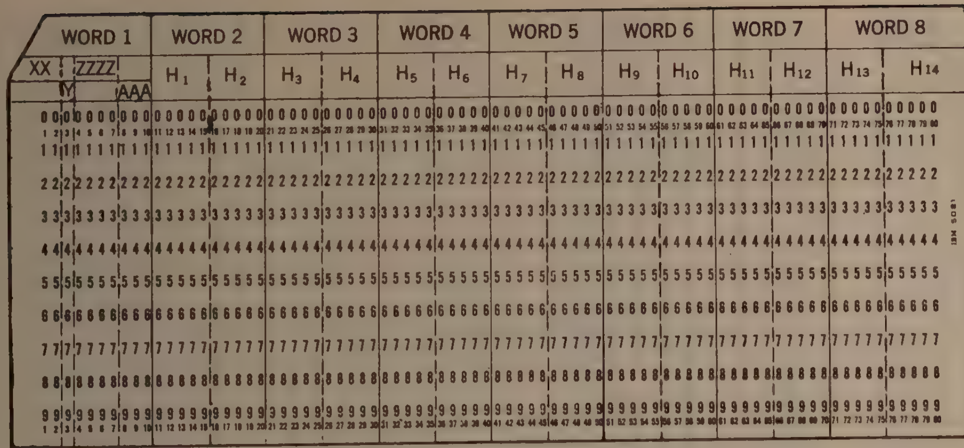
Y Specifies shift to left or number of zeroes to be added to the right when 5-digit-data fields in words 2 to 8, inclusive, on the card, are stored in regular 10-digit words in machine memory. The shift for flow values is variable in accordance with scaling desired. This applies to hydrograph and unit hydrograph "read-in," hydrograph punch-out, and routing-curve ordinates for level-pool routing. For all other cases it is as follows:

804 card, RE X U Gr, Shift is always 3
812 card, Musk Coeff C_1, Shift is always 7
813 card, Musk Coeff C_2, Shift is always 7
819 card, Hyd. total, Shift is always 4
809 card, Transfer, Shift is always 0
808 card, Transfer, Shift is always 0
817 card, Transfer, Shift is always 0
821 card, Transfer, Shift is always 0
810 card, Transfer, Shift is always 0

ZZZZ Serves as identification.

AAA For hydrographs and unit hydrographs "read-in" gives the location into which the first 5-digit-data field is to be read. Remaining data fields on card will go into the next consecutive higher memory locations. For programming reasons the actual machine memory locations are 400 higher than specified by "AAA" on the card. For a long hydrograph "AAA" for the first card is 001, for the second card 015, and so forth, by increments of 14. For a unit hydrograph the format is the same except that "AAA" equals 400 must be used to specify the number of terms in the unit hydrograph, then the first card will have "AAA" equals 401, second "AAA" equals 415, and so forth. When "AAA" is greater than 800 it identifies the location of the first instruction of a subroutine to which the program branches.

Hypothetical Problem.—Due to restrictions on the length of this paper, complete information on card formats on all routines will not be given. However, it is believed it will generally be evident from comparison of data given for the short hypothetical problem and the listing of input-data cards, given in Appendix A.



Hydrograph "Read-In"
(Length Assumed 72 Terms)

Cols. 1-10	11-15	16-20	----	76-80
	XXXXX.	XXXXX.	----	XXXXX.
14 Y ZZZZ 001	H1	H2	----	H14
14 Y ZZZZ 015	H15	H16	----	H28
14 Y ZZZZ *				
02 Y ZZZZ 071	H71	H72	Blank	

* 001 + multiple of 14

Rainfall Excess Times Unit Hydrograph

Data are entered in pairs, "time period" and "RE amount," with decimal point as shown, i.e., time period (abv. as "j") with units position in Cols. 14, 24, 34 ··· 74, and RE amounts in hundredths with decimal point in front of Cols. 19, 29, 39 ··· 79.

Cols. 1-10 XX 3 ZZZZ 804

	j		RE
Cols. 11-15	0XXX.0	Cols. 16-20	0XX.XX
21-25	0XXX.0	26-30	0XX.XX
71-75	0XXX.0	76-80	0XX.XX

FIG. 3.—CARD FORMATS

MUSKINGUM ROUTING

Columns	C_1 812 Card 00.XXX	C_2 813 Card 00.XXX	Q 802 Card * XXXXX.
1-10	XX7ZZZZ812	XX7ZZZZ813	XXYZZZZ802
11-15	$C_{1\text{-}a}$	$C_{2\text{-}a}$	I_0 †
16-20	$C_{1\text{-}b}$	$C_{2\text{-}b}$	$\bar{O}_0$ †
21-25	$C_{1\text{-}c}$	$C_{2\text{-}c}$	Q_a
26-30	$C_{1\text{-}d}$	$C_{2\text{-}d}$	Q_b
31-35	$C_{1\text{-}e}$	$C_{2\text{-}e}$	Q_c
36-40	$C_{1\text{-}f}$	$C_{2\text{-}f}$	Q_d
41-45	$C_{1\text{-}g}$	$C_{2\text{-}g}$	Q_e
46-50	$C_{1\text{-}h}$	$C_{2\text{-}h}$	Q_f
51-55	$C_{1\text{-}i}$	$C_{2\text{-}i}$	Q_g
56-60	$C_{1\text{-}j}$	$C_{2\text{-}j}$	Q_h
61-65	$C_{1\text{-}k}$	$C_{2\text{-}k}$	Q_i
66-70	$C_{1\text{-}l}$	$C_{2\text{-}l}$	Q_j
71-75	(Not	(Not	Q_k
76-80	Used)	Used)	Q_l

* "Y" Corresponds to scaling of flow values.

† Preceding inflow or outflow, if 99999 program assumes equal to I_1.

Note: Q value points are offset 2 data fields from corresponding C_1 and C_2.

Fewer than the maximum number of segments may be used to define curves as long as corresponding "XX" is used.

RESERVOIR-TYPE ROUTING

Columns	$\bar{O}$ 816 Card * XXXXX.	$(V+\bar{O})$ 805 Card * XXXXX.
1-10	XXYZZZZ816	XXYZZZZ805
11-15	$\bar{O}_a$	Prec. $\bar{O}$ **
16-20	$\bar{O}_b$	Prec. $(V+\bar{O})$ **
21-25	$\bar{O}_c$	$(V+\bar{O})_a$
26-30	$\bar{O}_d$	$(V+\bar{O})_b$
31-35	$\bar{O}_e$	$(V+\bar{O})_c$
36-40	$\bar{O}_f$	$(V+\bar{O})_d$
41-45	$\bar{O}_g$	$(V+\bar{O})_e$
46-50	$\bar{O}_h$	$(V+\bar{O})_f$
51-55	$\bar{O}_i$	$(V+\bar{O})_g$
56-60	$\bar{O}_j$	$(V+\bar{O})_h$
61-65	$\bar{O}_k$	$(V+\bar{O})$
66-70	$\bar{O}_l$	$(V+\bar{O})_j$
71-75	(Not	$(V+\bar{O})_k$
76-80	Used)	$(V+O)_l$

** If zeroes entered, program does a reverse table look-up to get starting $(V+\bar{O})$.

FIG. 4.—CARD FORMATS

Adapting Program to Other Basins.—Because the instruction deck is perfectly general, this program can be applied to any basin. A schematic diagram of the basin is required to indicate proper assembly of the input-deck data. Unit hydrographs for subareas in the basin will be part of this input-data deck as will be the reach-routing coefficients and the necessary control cards to guide the computation. The best guide to input-data deck arrangement is the study of specific examples. One is contained in a Missouri River Division pamphlet, which shows a complete manual computation, the corresponding input-data deck for the machine, and the output-data deck results for the July, 1951 flood for that portion of the Kansas River basin immediately upstream of Kansas City. Fig. 5 shows the Kansas River basin with reservoir and gage locations; Fig. 6 shows a portion of the Republican River basin, Harlan County Dam to Concordia, Kansas, divided into subareas of appropriate size for flow-determination computations; and Fig. 7 is a schematic flow diagram for the area shown in Fig. 6.

Computation Time.—A comparison between time required for machine and manual computation is of interest. The ratio is about 100 to 1 in favor of the machine. It should be kept in mind that this is for the computations only. Preparation of data is still a bottleneck requiring approximately the same amount of time for either manual or machine method.

PART II - SYSTEM RESERVOIR REGULATION PROGRAM

General Aspects of the Problem.—The computation procedures developed in the Missouri River Division office and described in Part I were combined with the computation procedures for regulation of a system of reservoirs through a flood accumulation and emptying cycle. With the resulting program, that is essentially complete, a system of reservoirs can, for the first time, be operated on the digital computer with reasonable consideration to sequential time periods and the various criteria normally applicable. By dividing the program into three parts, it will run on the IBM 650, but Part II runs at a rather slow rate. Part II also has been programmed for the IBM 704 computer and runs very satisfactorily. The entire program is also being prepared for the IBM 1620. Every effort has been made to generalize the program. It is believed that the operational program can be used on other basins with only a small portion of the programming effort (required as compared to that) required for the original effort.

The objective of the system reservoir-regulation program for flood control is to set reservoir releases so that insofar as practicable they will not contribute to damaging flows in downstream reaches. Not only do the individual reservoirs and flood events have to be considered by themselves, but also the status of the reservoir system as a whole in its ability to cope with occurrence of major floods that could possibly develop. That is, the effect of complete control of small floods must be weighed against the ability of the system to cope with major flood situations, and provision of high-release rates to evacuate storage must be balanced against the chance that these releases might contribute to damages in certain reaches if unforeseen storms occur after release have been made. Forecasts of streamflow at certain points in a river system have a basic roll in determining reservoir operation. It is apparent that any forecasts must be revised or updated as new information becomes available.

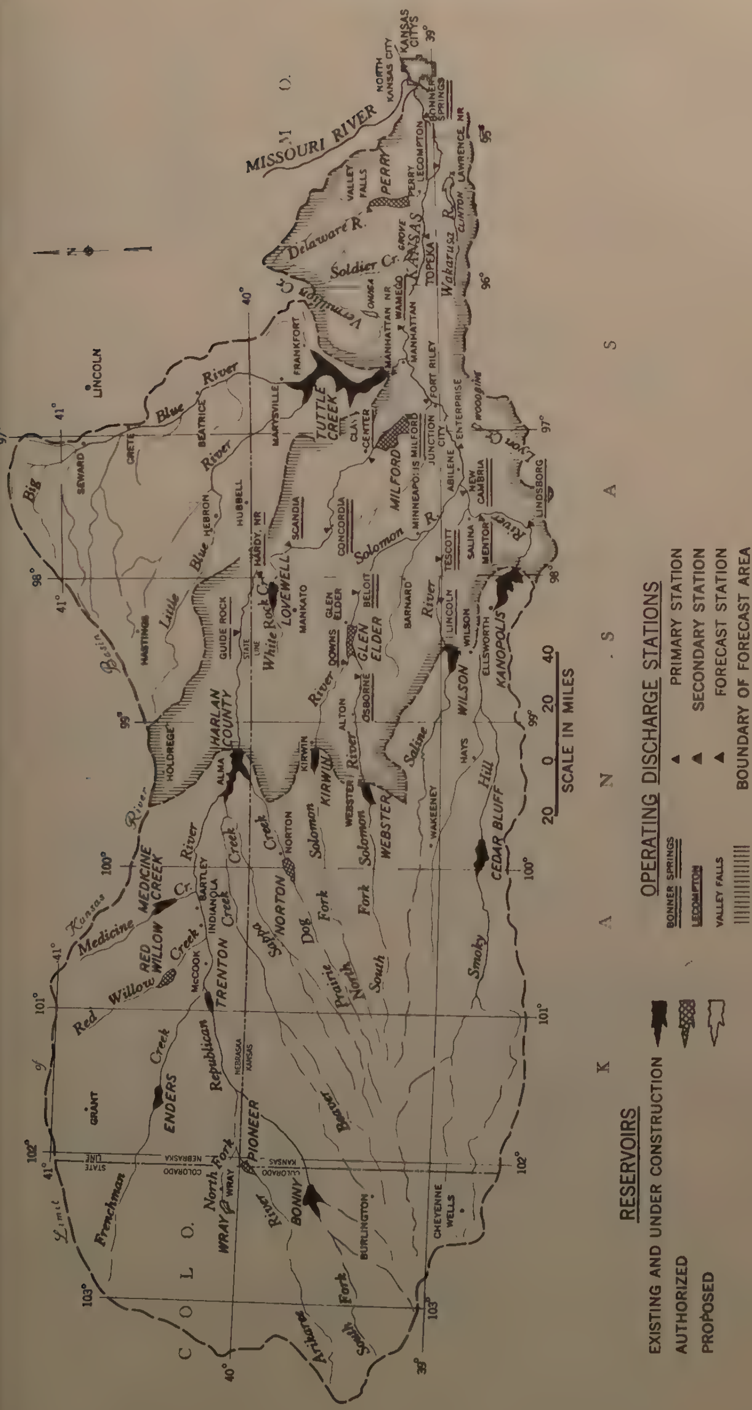


FIG. 5.—KANSAS RIVER BASIN - RESERVOIR AND GAGE LOCATIONS

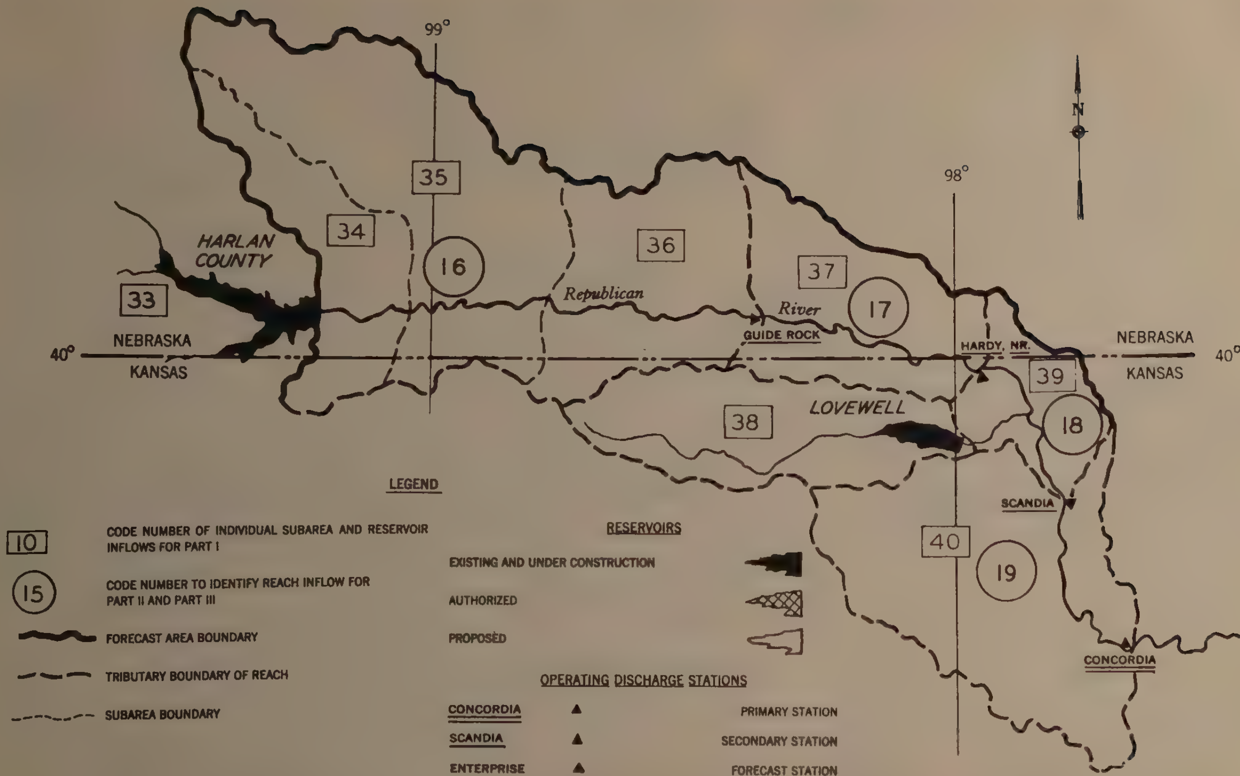


FIG. 6.—KANSAS RIVER BASIN - REPUBLICAN RIVER—HARLAN COUNTY DAM TO CONCORDIA, KANSAS STATIONS

Hydrologic Features.—The principal computation requirements listed subsequently were selected with due regard to minimizing the technical complications as much as possible and still accomplish the desired results. Details to be included in the first trail were of necessity dependent on the judgment of the engineer and are subject to improvement as computational experience is gained.

1. Initially, the system includes the seven lowermost reservoirs in the Kansas River basin and has sufficient flexibility to increase or decrease the number of reservoirs with a minimum of reprogramming.

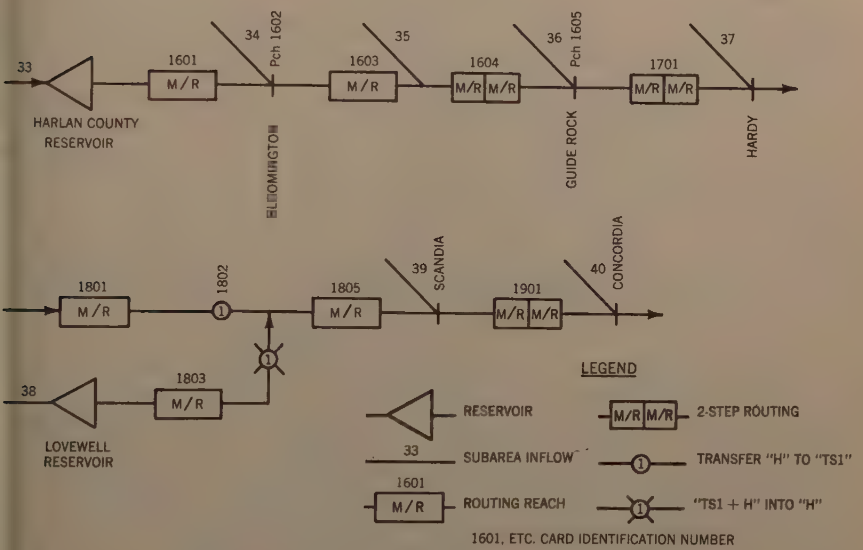


FIG. 7.—SCHEMATIC MAP OF REPUBLICAN RIVER BASIN - HARLAN COUNTY AND LOVEWELL RESERVOIRS TO CONCORDIA, KANSAS

2. Hydrologic computations are by 6-hr time periods. The basin downstream of the reservoirs is divided into 68 subareas averaging about 250 sq miles each (Fig. 8).

3. Forecasts of releases at each reservoir are computed by machine for each successive time period utilizing subarea runoff for 12 time periods in advance in lieu of rainfall forecasts. Data previous to the present time period are used in the forecasts as required.

4. The flood-control storage in each reservoir is divided into four zones. In addition, the surcharge zone above the flood-control pool and the conservation pool below are recognized. A limiting maximum release rate is provided as a constant for each zone.

5. Two limiting flow rates are utilized at each stream-gaging station as the basis for relating total system storage, reservoir release rates, and channel capacity.

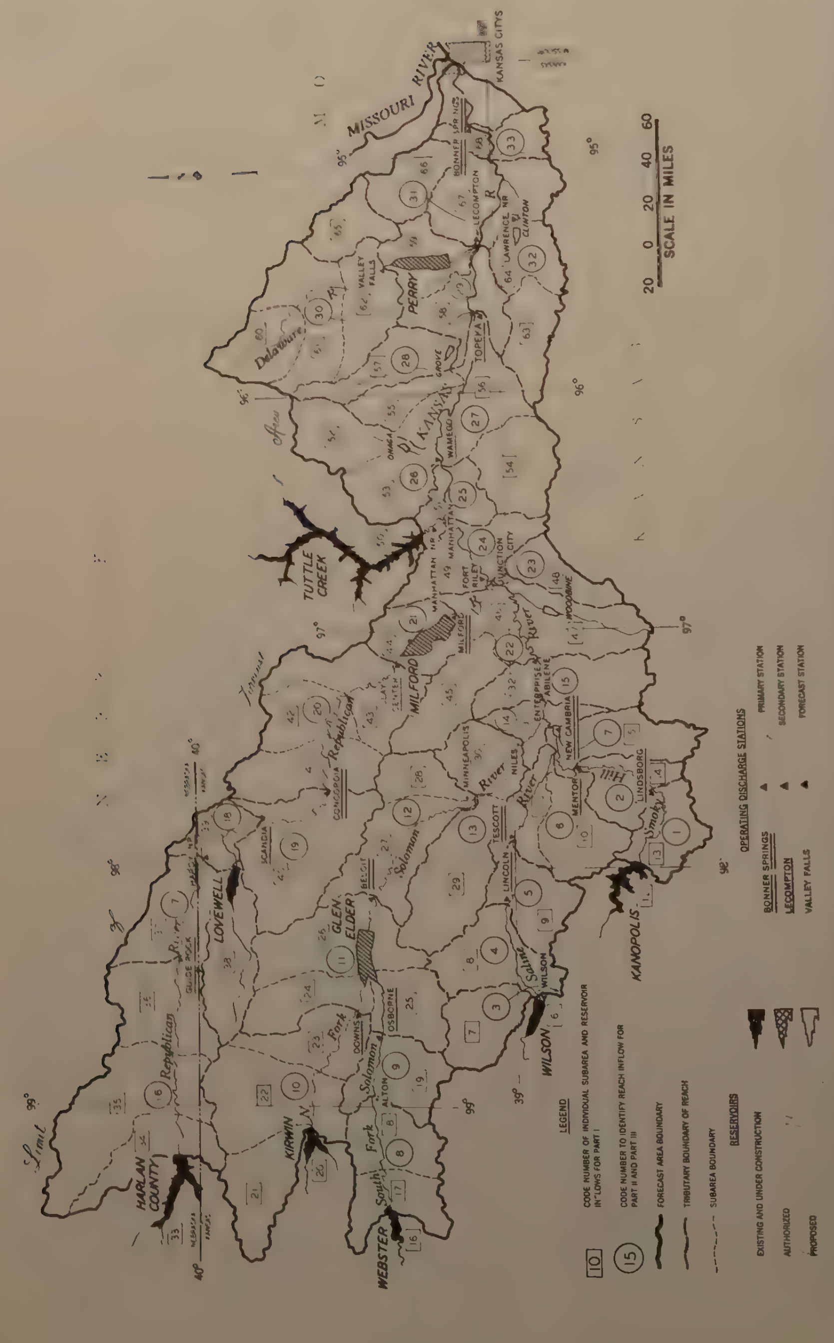


FIG. 2.—KANSAS RIVER BASIN, FORECAST ZONES - SUBAREAS AND REACHES

6. A method is provided to recognize, as input data, the need for reduced releases when floods are in progress downstream on the Missouri and Mississippi Rivers.

7. Reservoir outflow will be automatically reduced when floods are in progress, or forecasted to be in progress, on streams of the Kansas River basin downstream of the reservoirs.

8. Priorities for emptying reservoir storage are included.

9. The program is to run automatically through an entire flood and emptying cycle, with the exception that the three parts will be run separately.

10. Machine-data plotting is planned to minimize handwork. The immediate objective of the program and related studies is to refine the operational techniques for Tuttle Creek Reservoir, which was placed in full operation in 1961. The Kansas River basin has 13 reservoirs now in operation (1961), or under construction, and the system may eventually grow to approximately 26 reservoirs. Once the initial program is thoroughly proven, additional reservoirs will be incorporated into the studies as required. Later, the program with necessary modification, will be adopted to the Osage River basin in Kansas and Missouri where two reservoirs are under construction and seven additional reservoirs are authorized. The first combination of reservoirs selected for study includes seven reservoirs now in operation or under construction in the Kansas River basin as follows: Tuttle Creek, Kanopolis, Wilson, Kirwin, Webster, Lovewell, and Harlen County. The locations of these reservoirs are shown on Fig. 5 and physical data on Kansas River reservoirs in operation and under construction are given on Table 2.

The hydrologic computations, such as conversion of rainfall excess into runoff, hydrographs for individual subareas, and the flood routing to account for valley storage effects, are generally the same as described in Part I of this paper. The 17,000 sq-mile uncontrolled Kansas River drainage area is divided into 68 subareas. The subarea numbers, river stations, and their relation to the 7-reservoir system are shown on the map of the forecast area, Fig. 8. Discharge hydrographs have been computed at 24 stream-gaging stations, and streamflow ratings are prepared for 33 river reaches. In addition, the accumulated storage and the outflow have been computed for each reservoir. Aside from initial conditions and constants, rainfall excess by subarea, reservoir inflows, and downstream flood conditions constitute the input data.

Operational Procedures.—Reservoir regulation techniques developed in the regulation manuals for individual reservoirs are incorporated into the program in a general manner with some modifications so that techniques for all reservoirs will be as uniform as practicable. (The Corps of Engineers is responsible for regulation of flood-control storage in all Federally financed reservoirs. Reservoir regulation plans, usually referred to as regulation manuals, are developed for optimum accomplishment of the objectives established in the project authorizations and must be based on up-to-date hydrologic studies. Reservoir regulation manuals are prepared under the direction of the appropriate District Engineer. After approval by the Division Engineer and the Chief of Engineers, these manuals become the official operating plan for flood-control storage in the reservoir projects.) Generally, techniques for system operation, as provided in the master manual plans, override the requirements for individual reservoir operation. In some instances it was necessary to approximate rules for system operation in more detail than provided in the manuals. In order for the program to run automatically, all provisions

TABLE 2.—PERTINENT DATA ON

Line No. (1)	Name of project (2)	Location stream (3)	Miles above mouth (4)	Uncontroll present drainage area (sq. m (5)
1	Bonny	S. Fk. Republican R.	55	1,875
2	Pioneer	Arikaree River	7	1,911
3	Wray	N. Fk. Republican R.	24	169
4	Swanson Lake	Republican River	365	3,980
5	Enders	Frenchman Creek	52	2,240
6	Red Willow	Red Willow Creek	15	632
7	Harry Strunk Lake	Medicine Creek	9	838
8	Nelson Buck	Beaver Creek	--	1,405
9	Oberlin	Sappa Creek	108	1,073
10	Norton	Prairie Dog Creek	66	716
11	Harlan County	Republican River	236	8,561
12	Lovewell	White Rock Creek	16	354
13	Milford	Republican River	8	3,796
14	Cedar Bluff	Smoky Hill River	358	5,300
15	Kanopolis	Smoky Hill River	208	2,560
16	Wilson	Saline River	130	1,917
17	Kirwin	N. Fk. Solomon R.	67	1,363
18	Webster	S. Fk. Solomon R.	90	1,198
19	Glenn Elder	Solomon River	156	2,869
20	Woodbine	Lyon Creek	13	213
21	Angus	Little Blue River	--	1,163
22	Tuttle Creek	Big Blue River	12	9,556
23	Onaga	Vermillion Creek	13	287
24	Grove	Soldier Creek	25	142
25	Perry	Delaware River	5	1,117
26	Clinton	Wakarusa River	20	362

ANSAS RIVER BASIN RESERVOIRS

onstruction agency (6)	Status (7)	(Storage allocation (acre-feet) Flood control (8)	Other (9)	Total (10)
u of Reclamation	Completed	129,000	41,000	170,000
of Engineers	Authorized	87,000	28,000	115,000
u of Reclamation	Authorized	1,000	7,500	7,500
u of Reclamation	Completed	134,000	120,000	254,000
u of Reclamation	Completed	30,000	44,000	74,000
u of Reclamation	Under Constr.	50,000	38,000	88,000
u of Reclamation	Completed	52,000	39,000	91,000
u of Reclamation	Authorized	225,000	55,500	280,500
u of Reclamation	Authorized	135,000	31,000	166,000
u of Reclamation	Authorized	100,000	30,000	130,000
of Engineers	Completed	500,000	350,000	850,000
u of Reclamation	Completed	50,000	44,000	94,000
of Engineers	Under Constr.	700,000	380,000	1,080,000
of Reclamation	Completed	192,000	185,000	377,000
of Engineers	Completed	397,000	53,000	450,000
of Engineers	Under Constr.	510,000	245,000	755,000
of Reclamation	Completed	219,000	95,000	314,000
of Reclamation	Completed	200,000	72,000	272,000
of Reclamation	Authorized	722,000	120,000	842,000
of Engineers	Proposed	153,000	73,000	226,000
of Reclamation	Authorized	233,000	52,000	285,000
of Engineers	Under Constr.	1,933,000	413,000	2,346,000
of Engineers	Proposed	202,000	100,000	302,000
of Engineers	Proposed	104,000	53,000	157,000
of Engineers	Authorized	334,000	221,000	555,000
of Engineers	Proposed	255,000	129,000	384,000

for engineering decisions to be made as the flood progresses have to be reproduced by alternate procedures selected by the machine. Wherever possible, all constants such as pool storage, flow levels, and routing coefficients are placed in the program as constants that can be set at the beginning of a machine run and changed, if necessary, from time to time as the studies are made. By variation of the constants from run to run, the basis for exercising engineering judgment is presented and many more studies can be completed with the machine than by hand computation. Operational rules, assumed for initial studies, can be revised later if the necessity is indicated as the work progresses.

TABLE 3.—RESERVOIR PRIORITY, RESERVOIR OPERATION STATIONS, AND TIME PERIODS

Reservoir (1)	Priority (2)	Operation stations (3)	Six-hr time periods reservoir to station (4)
Tuttle Creek	1	Bonner Springs[a]	10
		Lecompton	6
		Topeka	5
		Wamego	2
Harlan County	2	Concordia[a]	12
		Scandia	10
		Hardy	8
		Guide Rock	6
Kanopolis	2	New Cambria[a]	11
		Mentor	7
		Lindsborg	3
Wilson	3	Tescott[a]	12
		Lincoln	6
Webster	3	Beloit[a]	10
		Osborne	4
Kirwin	3	Beloit[a]	10
		Downs	3
Lovewell	3	Milford[a]	12
		Clay Center	9
		Concordia	5
		Scandia	3

[a] Primary stations

In order to relate reservoir releases to streamflow forecasts downstrea as contemplated in the reservoir regulation criteria, a computational devic was developed that is called a "hold" in this description. The "hold" is automatically set by the machine for certain designated stations when compute forecasts exceed a previously established discharge limit.

"Holds" for the Missouri and Mississippi Rivers below the Kansas Rive are introduced as input data rather than by computations, as riverflow dat below the mouth of the Kansas River are not introduced into the computations

Priority in emptying reservoirs is based on the relation between unoccupie storage in the various reservoirs and the potential flood damage below th

reservoirs. The use of zones in the reservoirs provides the opportunity to set up various combinations for machine use in setting the individual reservoir releases for each time period. Reservoir priorities initially established and time of water travel in 6-hr time periods are shown in Table 3.

Recognition is given to the system operation criterion of using more of the available channel capacity for reservoir releases in proportion to the system storage occupied by setting two levels of channel flow, not to be exceeded at each station. The lower level of channel flow is used when flood-storage capacity of the system is less than 50% occupied and the higher level of flow is used when system flood-storage capacity is more than half full.

Streamflow forecasts are computed for each time period as a part of the routine necessary to set the reservoir releases. The forecasts are designed to reasonably approximate the overall value to system operation of forecasts based on reported data from the 17,000 sq miles of uncontrolled Kansas River basin drainage area. The forecast requirement greatly complicated the program as the entire forecast routine has to be performed for each time period and the reservoir releases established before going to the next time period. This is basically different from the data-handling procedure for the flow-determination program described in Part I of this paper. The entire computation for Part I progresses consecutively from period to period through the entire flood period for each subarea and station involved before going to the next, with a minimum of data in the machine at any one time period. Forecasting procedures were designed individually for each station but generally utilize discharge from the reservoirs and for stations nine periods previous to the period under consideration and runoff from the subareas twelve periods in advance. The actual runoff twelve periods in advance was adopted as a compromise between computational expediency and the combination of rainfall forecasts and recessional flow normally utilized by a forecaster in actual operation. The hydrographs thus computed at the key stations by streamflow-procedures are without the effects of reservoir releases for the period under consideration and for future periods. Hydrographs can be compared with previously determined limiting flows at the key stations to determine whether reservoir releases are in order and the maximum release permissible at the various stations.

In order to utilize the discharge station forecasts, the stations are classified as "Primary" and "Secondary." Primary and secondary stations are listed in Table 3 and are shown graphically on Figs. 5 and 8. The entire forecast is scanned at the primary station to establish "holds" as required. "Holds" are related to reservoir-pool levels by zones through stored data tables. When a "hold" is applicable the reservoir release is automatically set at a minimum. Forecast streamflows at secondary stations, usually less than twelve 6-hr time periods of flow downstream of individual reservoirs, are utilized to set the actual reservoir releases once such releases are indicated as permissible by the lack of applicable "holds."

Although flood-control storage of Kansas River basin reservoirs is generally set to contain the maximum flood of record, routines are included in the program for each reservoir to operate through a flood exceeding the design capacity. During surcharge operations reservoir releases are based on an accumulated storage versus outflow relationship, which is obtained from surcharge operation criteria given in the regulation manual. One exception is that the maximum release for the top zone of the flood-control pool is passed

when the reservoir is in the surcharge zone, if channel capacities downstream will not be exceeded. The surcharge operation can be changed by inserting in the appropriate memory location a different storage versus outflow table at the beginning of the run.

Minimum reservoir releases have been established at each reservoir and are used when no flood releases are permissible, or when all flood storage has been released. Kanopolis reservoir has an uncontrolled outlet and minimum releases are established for this project on the basis of accumulated storage.

The end results obtained from the program are as follows:

1. Reservoir inflow, accumulated storage, and outlfow by 6-hr periods.
2. Computed natural discharge hydrographs by 6-hr periods at all discharge stations downstream of the reservoirs to the mouth of the Kansas River. The actual discharge hydrograph can be added for comparative purpose if desired.
3. Computed natural hydrographs modified by reservoir operation. All hydrographs at each station are normally plotted on the same sheet. Likewise, the inflow accumulated storage and outflow hydrograph can be plotted on the same sheet. Hydrographs for different trial runs can be most readily compared if plotted to the same scales.

Provision is made in establishing the card input and final output for plotting hydrographs of discharge and storage data on the IBM 407 accounting machine. With minor modification other forms of digital plotting could be utilized.

Programming Techniques.—The system operation program is logically divided into three parts and, at least in the initial phase, it has been found more expedient to take advantage of this feature. Part I converts the rainfall excess by 6-hr periods to subarea hydrographs and combines and routes subarea hydrographs to the point at which flow may be affected by reservoir operation. This part of the program also derives the computed actual hydrograph that is compared later with the hydrographs modified by reservoir regulation. Part II includes all the routines necessary to establish the reservoir releases for each time period using the subarea discharge hydrographs computed in Part I. Part III of the program uses the results of Part I and Part II and computes the modified hydrographs at the key stations which are then plotted on the same sheet to the same scale with the computed actual hydrograph, by use of the IBM 407 accounting machine, to show the system reservoir effects. The advantage of handling the problem in three parts is that Part I and Part III, that are basically the flow determination routines described in Part I of this paper, can be worked on a smaller machine such as the IBM 650 or IBM 1620. Part I of the program can be processed and thoroughly reviewed before proceeding to Part II, which can normally be run more economically on a large-size computer. It is recognized that the economics for any particular organization may depend on the type of equipment available, work loads, programming experience, and other factors.

Basin flow schematic diagrams for Part I and Part III of the system operation program for the Kansas River basin are similar in construction to Fig. 7, which is the flow-determination program for a portion of the Republican River basin described previously. Computational routines for Part II of the program are shown in general block form on Fig. 9, and the routines are described more fully in Appendix B. The relation between subareas for Part I and reaches for Parts II and III is shown in Appendix C. The sequence of computational

steps for computing the forecasts and setting reservoir releases for each time period is shown in the flow-diagram form in Fig. 10. This flow diagram is an expansion of Subroutine 50 shown in Fig. 9. Part II of the program is written in Fortran for the IBM 704 computer and is made sufficiently general that the system of reservoirs to be studied can be reduced or expanded by an engineer or programmer familiar with the program and computational pro-

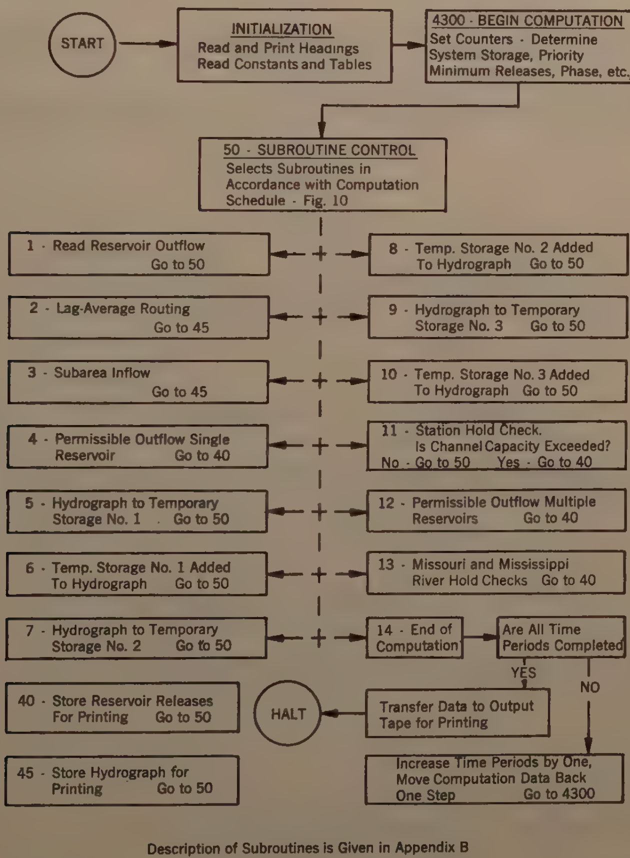


FIG. 9.—DIAGRAM FOR FORECASTS AND RESERVOIR RELEASE COMPUTATIONS - PART II

cedures utilized. The detailed computations are too voluminous to illustrate in a paper of this scope.

Basic data and final output of all parts of the program are on IBM-type cards. This is advantageous to plotting final hydrographs on the IBM 407 accounting machine. Prior to actual plotting, the output cards must be specially processed by a digital computer, such as the IBM 650, to obtain the plotting

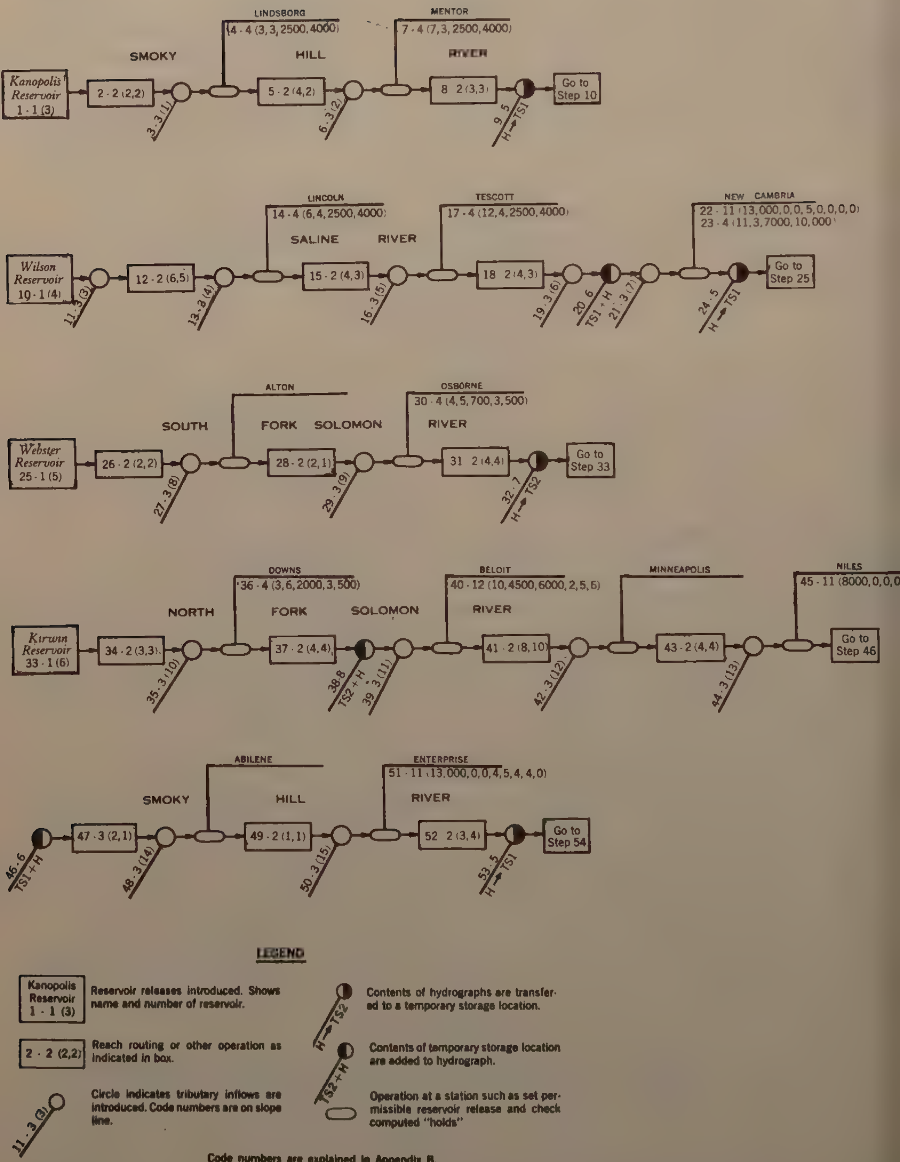


FIG. 10.—KANSAS RIVER BASIN—SUBROUTINE

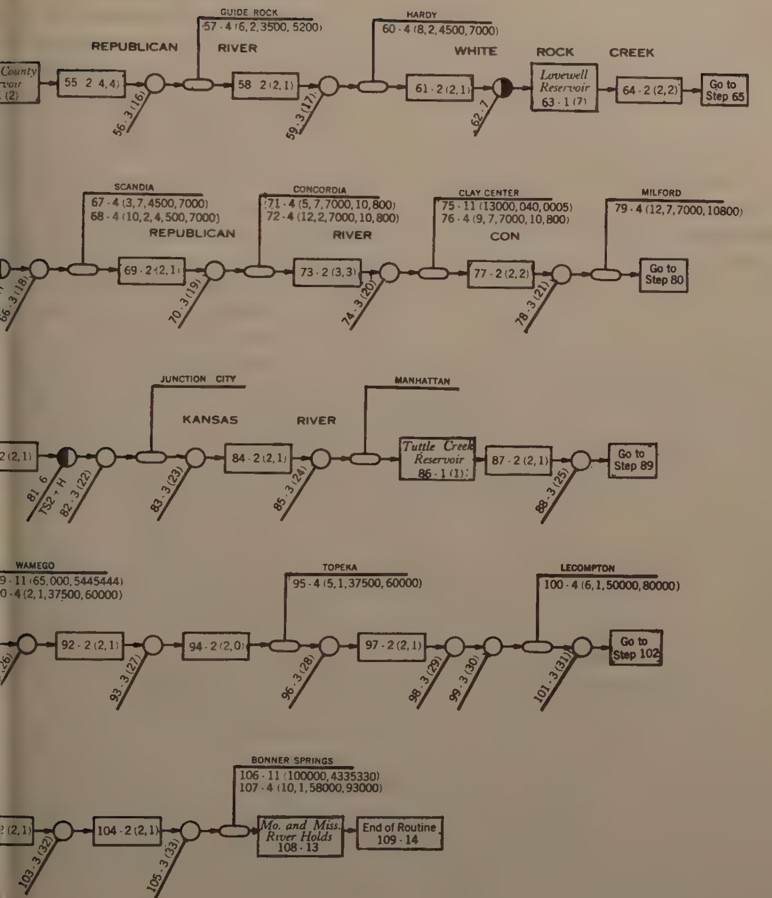


FIG. 10. - KANSAS RIVER BASIN SUBROUTINE SCHEDULE TO SET RESERVOIR RELEASES

SCHEDULE TO SET RESERVOIR RELEASES

cards. In effect, the plotting is accomplished by using each of the 120-column spaces on the machine as a time period. The spaces normally occupied by lines of figures are utilized as increments of discharge or vertical ordinates of the hydrograph and may extend across more than one sheet of paper if desired. A special symbol can be used for half-space plotting, thus making the plotting increments per inch double the number of lines that are to be presented. The digital computer sorts the cards in such a manner that all plotting across the sheet is done a line at a time, starting from the top. The digital computer also indicates the symbol to be used for plotting so that the points for more than one hydrograph can be plotted simultaneously. The points are later connected manually with lines if desired. Final plotted hydrographs are on sheets 14 in. wide and multiples of 11 in. in height. The plotting is greatly enhanced by the use of specially prepared 10 x 10 to-the-inch cross section paper, which has been developed and is now available commercially. Although this method of plotting is not ideal, in that considerable time and expense as well as trained technicians are involved, it has been found to be definitely more economical and much faster than manual procedures. Further, it assures a uniform product and affords a practicable method of assimilating the formidable bulk of data utilized in analyzing an extensive system of reservoirs. There has been no opportunity, as yet, to compare this method of plotting with other electronic plotters available. In this case also, the most economical procedures may be dictated by availability of equipment and experience of personnel.

Engineering Applications.—The program of the reservoir-operations system described previously has many applications. For the Kansas River basin system of reservoirs, the first application will be to develop and refine regulation criteria for the Tuttle Creek Reservoir simultaneously with further refinement of regulation criteria for the presently existing system of reservoirs. It will be further expanded to include other reservoirs to be added to the lower basin, such as Milford and Perry, now in the active design status. It may be found practicable to add upstream reservoirs, such as the four upstream of Harlan County, so the entire basin can be operated as a single unit for the analysis of the system operation for actual or transposed regional storms. Similar applications are believed to be practicable for reservoir systems in other basins. Likewise, the program appears to be applicable to further adjust and refine system reservoir-regulation techniques on the basis of operational experience. As detailed knowledge or changes become available on such factors as channel capacities, effects of local inflow, relation of reservoir releases to blocking of local drainage and bank covering, and economic evaluation of reduction in peak stages versus increased duration of high-inbank flows, the system operation during past floods can be restudied with a nominal expenditure of time and money.

The initial preparation of hydrologic data is a rather laborious process for storm periods such as experienced over the Kansas River basin in 1951. The sequence of storms began in April, 1951 and continued through July 13. There was a moderate flood in September, 1951 and reservoir emptying continued through December. Thus, the entire period of operation to be investigated totals 230 days. However, once the 6-hr rainfall data by subareas and the actual stream-discharge data at the key stations are established and transferred to cards, it can be utilized repeatedly with relatively small expenditures of time or cost. It is presently planned to investigate the six most severe floods on the lower Kansas River during the period 1930 to date. Like-

wise, streamflow routing coefficients, once developed, usually can be used for all floods with little modification.

Electric Flood Model.—The day-to-day operation of a system of reservoirs upstream of potential damage centers, such as the Kansas Citys where the 1951 flood alone caused $500,000,000 damage, should include a method for quickly obtaining the optimum reservoir gate setting (or outflow in cubic feet per second) for each new situation and through the emptying cycle as a flood develops. This requires many trial computations of reservoir operation. Digital computing afforded the opportunity for a very definite improvement in this field and is being applied to system-reservoir operation as described in this paper. However, the limited number of trials that can be accomplished in one working day and the cost of equipment of a size to expeditiously handle the computations severely limit the effectiveness of digital computing for setting reservoir releases on a day-to-day basis. For this reason, the Kansas City District office of the Corps of Engineers accepted a proposal by the University of California to develop an electric flood model on a Research and Development Contract that will perform the rainfall conversions and flood routing associated with system-reservoir operation in a fraction of a second. It is planned to incorporate 12 or 14 existing and authorized reservoirs located in the lower Kansas River basin into the model. Development of the model began in 1959, and it is hoped to have it completed and in operation during the summer of 1961.

The basic principles of the electric flood model are based on the close similarity between the differential equations governing the actions of water and electricity as follows:

Water	Electricity
Discharge	Current
Elevation of water surface	Potential or voltage
Storage volume	Charge on a capacitator

Although there were many difficulties to be overcome because of the strongly nonlinear character of open channel hydraulic systems, James A. Harden, M. ASCE, has developed electronic analog elements that satisfactorily duplicate the ability of channels to transmit and attenuate flood waves. In addition, convenient means of application of rainfall excess and channel-storage effects were developed for conversion of rainfall excess to discharge hydrographs, simulate streamflow routing, and display the resulting hydrographs in the form of a standing wave on an oscilloscope. Electric analogs for open channel flow were first investigated by H. A. Einstein, F. ASCE, and Harder of the University of California, Berkeley, Calif., in connection with an investigation of the Delta Regions about 60 miles east of San Francisco.[6]

Before entering into the contract for the electric flood model, careful consideration was given to the use of digital computers for day-to-day reservoir operation; but the electric flood model has some worthwhile advantages, particularly in the estimated time for translation of assembled hydrologic data for a system of reservoirs into terms of forecast streamflow hydrographs at the damage points, as well as pool elevations from assumed reservoir operation. With rainfall excesses and a particular scheme for gate operation for a 10-day period on cards or tape ready to go, it would probably be

6 "An Electric Analog Model of a Tidal Estuary," by H. A. Einstein and J. A. Harder, Proceedings, ASCE, Vol. 85, No. WW 3, September, 1959.

possible to have the results in tabular form ready for use with a medium-size digital computer readily available, after an hour or so. The reports for a second set of assumptions presumably could be made available at the end of another hour. Thus, a relatively small number of trials can be made during any particular working day.

The electric flood model, on the other hand, once the rainfall excesses and reservoir releases are set, goes through the computation of an entire flood sequence for 40 time periods in 1/200 sec, continuously repeating the operation. The measuring units can continuously show the end results; for instance, the time history of either stages or discharges at any given downstream station in the Kansas River. This hydrograph can be read from an oscilloscope or photographed for future reference. Adjustments can be made in the gate settings at a particular reservoir by resetting pins and the results of the change noted on a downstream station immediately. Various rainfall forecasts can be entered and the results observed at all reservoirs and downstream stations as quickly as the oscilloscope is switched from one station to another. It is believed that this model will be very effective in combining hydrographs to obtain the stream hydrographs from subareas, the flow of water through reservoir outlets, and the effects of valley storage on streamflow. It has also been possible to control the number of times that rainfall excesses and gate openings are effective and the timing on downstream discharges as in digital computing.

The electronic units, except the oscilloscope, will be completely transistorized, and all timing will be controlled electronically. The entire unit, if assembled in one unit, will have approximate overall dimensions (less shelves and removable projections) about 60 in. in height, 72 in. in length, and 30 in. in depth, and is expected to weigh about 1,500 lb. Design and testing of the circuitry was completed in August 1960, and the University of California is now in the process of constructing sufficient units to assemble the complete model.

Rainfall excesses, reservoir releases, and antecedent flow conditions are the three elements of basic data that are supplied to the model in actual operation. Rainfall excesses for each subarea by 6-hr periods are entered into the model by use of small pins. The area required for all possible combinations of pins is rather large, but only a small portion will be in use at any one time. It is planned, also, to set reservoir releases over each 6-hr period by use of pins. Reservoir releases over the entire range from zero flow to the peak spillway-design discharges can be set to 200 cfs increments, if desired. It is planned to make provisions to move all settings forward from day-to-day by switching arrangements that eliminate the necessity of resetting all rainfall-excess and reservoir-release values to be retained in the model, that will normally take care of antecedent conditions. The final model will incorporate a maximum of 14 reservoirs, 56 subareas, and about 30 stage and discharge stations. Some flexibility will be possible in that future reservoirs can be added to the system as they are constructed and placed in operation.

CONCLUSIONS

Both the flow determination and System Reservoir Operation programs are good examples of the advantageous use that can be made of present-day digital computers, now easily available for use in hydrologic engineering studies. The flow-determination program has been tested by several-years' usage by the

Corps of Engineers in the Missouri River Division and its district offices. However, as the program instructions are perfectly general, they can be applied to any river system. The physical characteristics of the basin under study are incorporated in the unit graphs and routing tables used in the input-data deck for the particular system considered. The system reservoir-operation program should be regarded as being in a preliminary form and subject to substantial improvement as it is tested in actual problems. Hydrologic procedures incorporated into these programs are currently in use in many engineering offices but the application of digital computing to perform these computations is relatively new and many possibilities remain to be explored.

Extension of the flow-determination program into the system reservoir-operation program of this type is believed to be an entirely new application. It is expected to be of practical use for analysis of the effect of the reservoir system for past floods of record.

The electric flood model of the lower Kansas River basin is an electronic analog-type computer and gives promise of being a very useful tool in the day-to-day operation of a complex system of reservoirs for flood control. It is also a new approach in the field of reservoir regulation and in due time will be tested by actual operating experience.

ACKNOWLEDGMENTS

The writers are indebted to Nicholas Barbarossa, F. ASCE, Chief, Hydraulic Section, United States Army Engineer Division, Missouri River, Omaha, Nebr., for valuable consultation and encouragement as the flow-determination and system-operation programs were developed; and to Otto T. Steiner, Chief, Programming Sub-Section, United States Army Engineer Division, Missouri River, Omaha, Nebr., mathematician, for programming the computational procedures and adaptations to the various digital computers.

APPENDIX A.—HYPOTHETICAL PROBLEM

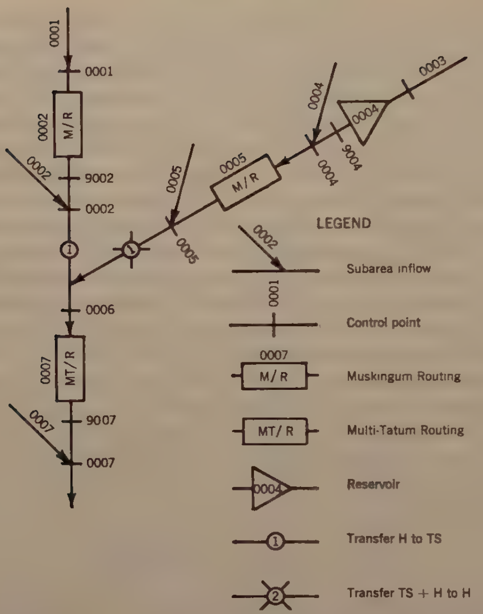


FIG. 11.—SCHEMATIC MAP FOR HYPOTHETICAL PROBLEM

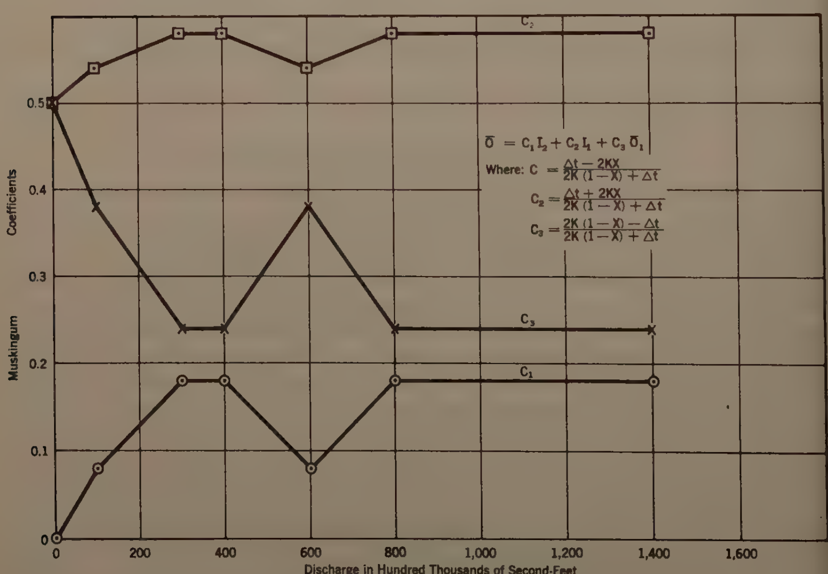


FIG. 12.—MUSKINGUM ROUTING COEFFICIENTS

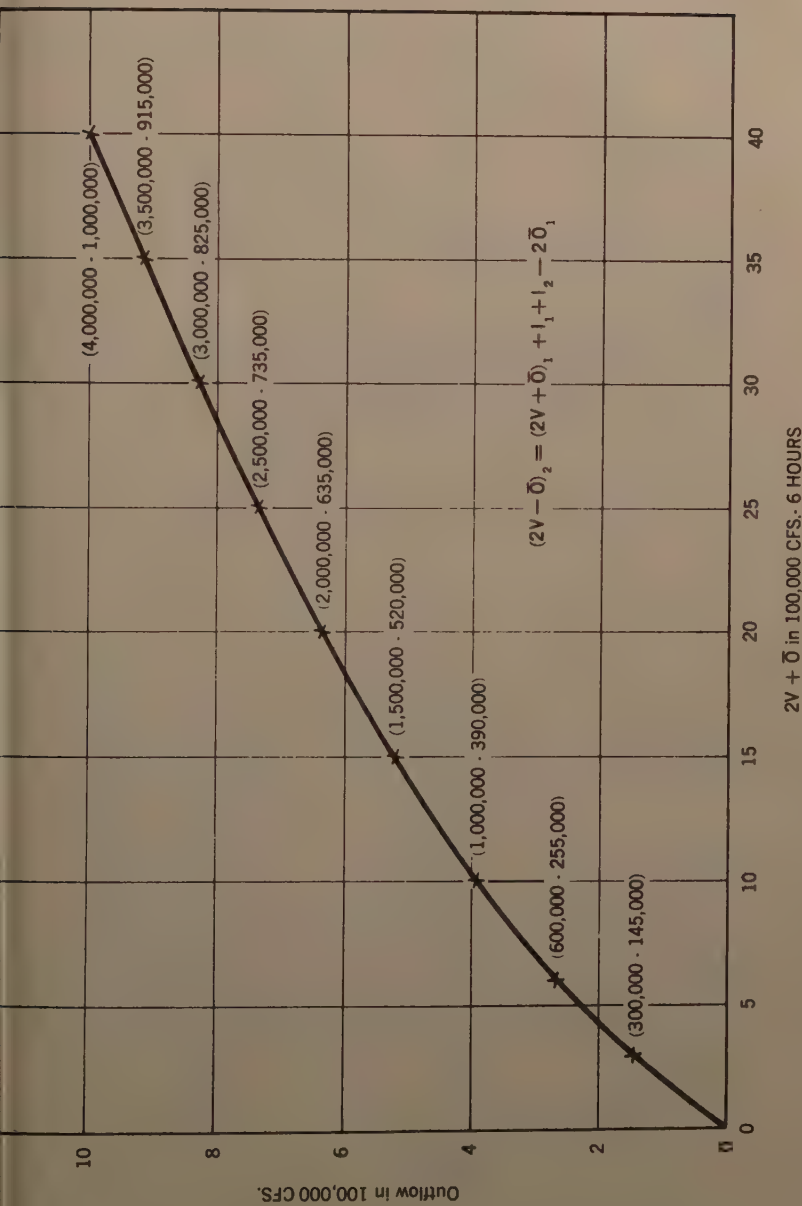


FIG. 13.—LEVEL-POOL RESERVOIR ROUTING

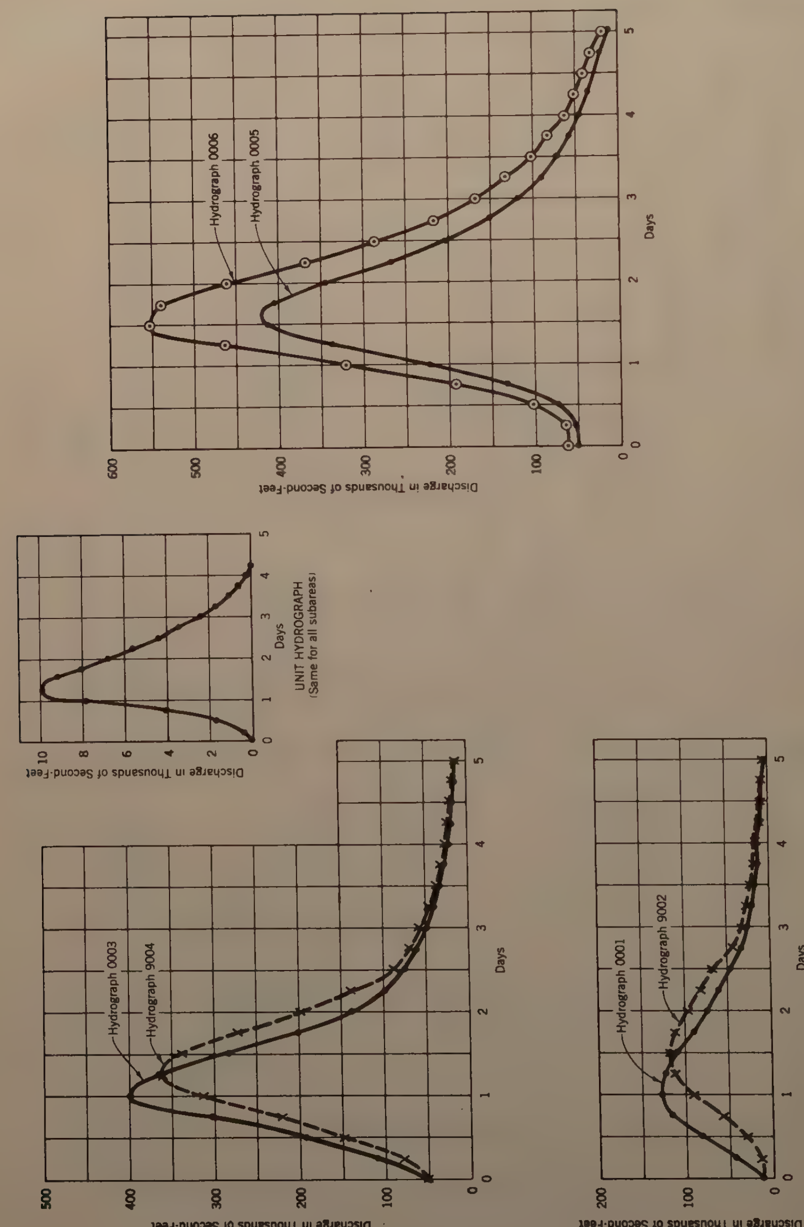


FIG. 14.—HYDROGRAPHS FOR HYPOTHETICAL PROBLEM

(*a*) **Sheet 1 of [illegible]**

```
00 [illegible] 0000 815 0000210000

14 2 0001 001 0013000440 0082001170 0128001240 0110000900 0074000600 0048000330 0027000200
07 2 0001 015 0017000130 0017000100 0009000090 0004300000

07 7 0002 812 0000000080 0018000180 0008000180 0018000000
07 7 0002 813 0050000540 0058000580 0054000580 0058000000
09 3 0002 802 9999999999 0000000100 0030000400 0060000800 0140000000

00 2 9002 811

01 4 0002 400 0001600000
14 1 0002 401 0004000170 0041000790 0100000920 0080000680 0056000440 0034000240 0017000100
02 1 0002 415 0006000020

04 3 0002 804 0002000050 0004000200

00 2 0002 811

02 4 0002 819 0000000021

00 0 0000 809

14 2 0003 001 0050001100 0192003030 0400003650 0283002000 0138000980 0075000610 0049000400
07 2 0003 015 0033000280 0022000190 0017000150 0013000000

10 3 0004 816 0000000145 0025500390 0052000635 0073500825 0091501000
12 3 0004 805 0000000000 0000000300 0060001000 0150002000 0250003000 0350004000

00 2 9004 811

01 4 0004 400 0001600000
14 1 0004 401 0004000170 0041000790 0100000920 0080000680 0056000440 0034000240 0017000100
02 1 0004 415 0006000020

04 3 0004 804 0002000200 0003000400

00 2 0004 811
```

(*b*) **Sheet 2 of 2**

```
07 7 0005 812 0000000080 0018000180 0008000180 0018000000
07 7 0005 813 0050000540 0058000580 0054000580 0058000000
09 3 0005 802 9999999999 0000000100 0030000400 0060000800 0140000000

01 4 0005 400 0001600000
14 1 0005 401 0004000170 0041000790 0100000920 0080000680 0056000440 0034000240 0017000100
02 1 0005 415 0006000020

04 3 0005 804 0002000150 0003000150

00 2 0005 811

02 4 0005 819 0000000021

00 0 0000 808

00 2 0006 811

02 4 0006 819 0000000021

06 2 0007 806 0400005000 1250037500 3750012500
05 2 0007 806 0300002000 2500050000 2500000000
06 2 0007 806 0200000500 1250037500 3750012500
07 2 0007 806 0100000000 0620025000 3760025000 0620000000

00 2 9007 811

01 4 0007 400 0001600000
14 1 0007 401 0004000170 0041000790 0100000920 0080000680 0056000440 0034000240 0017000100
02 1 0007 415 0006000020

04 3 0007 804 0002000150 0003000150

00 2 0007 811

02 4 0007 819 0000000021
```

FIG. 15.—LISTING OF INPUT-DATA CARDS FOR HYPOTHETICAL PROBLEM

```
14 2 9002 001 0013000133 0029600585 0092401138 0119001115 0096600816 0067900555 0042500337
07 2 9002 015 0026200212 0017000169 0013300111 0004300000 0000000000 0000000000 0000000000

14 2 0002 001 0013000135 0030400613 0099701270 0139401355 0118401004 0083700684 0052500414
07 2 0002 015 0031500249 0019100181 0013700111 0004300000 0000000000 0000000000 0000000000

00 0 0002 819 0000012073 0000012073 0000000000 0000000000 0000000000 0000000000 0000000000

14 2 9004 001 0050000790 0147702209 0311803595 0335502721 0198001375 0088200687 0055500449
07 2 9004 015 0036800307 0025200207 0018100161 0014100000 0000000000 0000000000 0000000000

14 2 0004 001 0050000798 0152702359 0344004111 0393903249 0243601759 0119400931 0073900579
07 2 0004 015 0045600359 0028000215 0018100161 0014100000 0000000000 0000000000 0000000000

14 2 0005 001 0050000518 0073401318 0222803370 0412604049 0345402691 0202601496 0115700911
07 2 0005 015 0071900568 0044800350 0027500224 0014100000 0000000000 0000000000 0000000000

00 0 0005 819 0000031301 0000031301 0000000000 0000000000 0000000000 0000000000 0000000000

14 2 0006 001 0063000653 0103801931 0322504640 0552005404 0463903695 0286302180 0168201325
07 2 0006 015 0103400817 0063900531 0041100335 0018400000 0000000000 0000000000 0000000000

00 0 0006 819 0000043375 0000043375 0000000000 0000000000 0000000000 0000000000 0000000000

14 2 9007 001 0004700224 0051700923 0175103126 0443205156 0525604709 0377302900 0226601845
07 2 9007 015 0150701197 0094000742 0059700497 0041000000 0000000000 0000000000 0000000000

14 2 0007 001 0004700230 0054901010 0193103394 0472005414 0547804895 0392303017 0235301907
07 2 0007 015 0154701212 0094000742 0059700497 0041000000 0000000000 0000000000 0000000000

00 0 0007 819 0000044813 0000044813 0000000000 0000000000 0000000000 0000000000 0000000000
```

FIG. 16.—LISTING OF OUTPUT-DATA CARDS FOR HYPOTHETICAL PROBLEM

APPENDIX B.—DESCRIPTION OF OPERATIONAL CODES SHOWN ON FLOW DIAGRAM, FIG. 10.

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1. The first position in each code is the step number in the forecasting routine and is designated as position A in subroutine descriptions.

2. The second position in each code is the subroutine number that varies from 1 to 14 and also corresponds to subroutine numbers on Fig. 9. Numbers following the subroutine number are operational data that vary with the subroutine as explained below.

SUBROUTINES

No. 1 - Read Reservoir Outflow.—Transfers reservoir outflows to the proper working space for the number of past periods used to compute downstream forecasts. Format of code: A - 1(B), where B is the reservoir number.

No. 2 - Lag-Average Routing.—Routes flows using lag-average procedure from upstream to downstream end of river reach, usually to the next downstream station. Format of code: A - 2(C, D), where C is the number of periods to be averaged and D is the number of periods to be lagged.

No. 3 - Read Tributary Inflow.—Reach inflows are introduced for the assumed number of known periods, in this case 9, prior to the present period and 12 in advance of the present period. Subareas comprising each reach and reach identifications are shown in Appendix C. Format of code: A - 3(E), where E is the reach number.

No. 4 - Permissible -Outflow - - Single Reservoir.—Reservoir outflow for the present time period is set as the smaller of the following:

(1) The difference between forecast flow for the controlling time period and the maximum permissible controlled flow at this station, or;

(2) The maximum flow permissible at dam or at a station between the dam and this station already computed. Format of code: A - 4(F, G. HHHHH, IIIII), where F is the controlling period; that is, number of time period the station is below reservoir; G is the reservoir number; HHHHH is the permissible maximum controlled flow in second-feet when the system is in Phase I; and, IIIII is a similar value when the system is in Phase II.

No. 5 - Hydrograph Discharge Values Transferred to Temporary Storage No. 1.—Accumulated discharge values in the hydrograph working space are transferred to temporary storage location No. 1. Format of code: A - 5.

No. 6 - Temporary Storage in No. 1 Added to Hydrograph.— Discharge values previously transferred to temporary storage location No. 1 are transferred back to the hydrograph working space and are added to the discharge values now in corresponding locations in the hydrograph space. Format of code: A - 6.

No. 7 - Hydrograph Discharge Values Transferred to Temporary Storage No. 2.—Same as subroutine No. 5 except that hydrograph discharge values are transferred to temporary storage location No. 2. Format of code: A - 7.

No. 8 - Temporary Storage in No. 2 Added to Hydrograph.—Same as subroutine No. 6 except that values in temporary storage location No. 2 are transferred back to the hydrograph. Format of code: A - 8.

No. 9 - Hydrograph Discharge Values Transferred to Temporary Storage No. 3.—Same as subroutine No. 5 except that hydrograph discharge values are transferred to temporary storage location No. 3. Format of code: A - 9.

No. 10 - Temporary Storage in No. 3 Added to Hydrograph.—Same as subroutine No. 6 except that values in temporary storage location No. 3 are transferred back to the hydrograph. Format of code: A - 10.

No. 11 - Primary Station Hold Check.—Checks forecast hydrograph at this point for all forecast time periods (exclude periods previous to present period), against specified maximum flow. If specified maximum flow is exceeded, sets the previously computed release for each reservoir indicated in the parameter figures to minimum if pool level is in a zone for which the hold is applicable. If hold is not applicable the reservoir release remains at the level established by subroutine No. 4. Format of code: A - 11(JJJJJ, K, L, M, N, O, P, Q), where JJJJJ is the specified maximum flow; K through Q are the highest zone in the reservoirs for which the "hold" is effective. The reservoir is indicated by the position, that is, K is reservoir No. 1, which is Tuttle Creek. A zero in the reservoir position indicates that the reservoir is not considered. For example, the values of K to Q of 0,0,5,0,0,0,0,0 indicates that reservoir No. 3, Kanopolis, is to be reduced to minimum outflow for all levels, Zone 5 and below. In this code reservoir zones are identified as follows:

Zone	Code No.
Conservation pool	1
Buffer zone, bottom of flood-control pool	2
Zone 1 of flood-control pool (lowest)	3
Zone 2 of flood-control pool	4
Zone 3 of flood-control pool	5
Surcharge zone	6

No. 12 - Permissible Outflow - Multiple Reservoirs.—Reservoir outflow for the present time period for each of two or more reservoirs sharing the channel capacity at a single control station are set so as to allocate the available channel capacity at this station among the reservoirs included on the basis of previously computed priorities and programmed schedules for priorities. Releases established in this manner are subject to reduction to minimum, if indicated by "hold" checks and, in any event, may not exceed permissible rates at the respective reservoirs. Format of code: A - 12(RR, SSSSS, TTTTT, U, V, W), where RR is the number of time periods of water travel for selection of the exact forecast period; SSSSS is the channel capacity in second-feet for Phase I; TTTTT is the channel capacity for Phase II; U is the number of reservoirs included; and, V and W are the reservoir identification numbers.

No. 13 - Missouri and Mississippi River Hold Check.—If there is a Missouri or a Mississippi River "hold" indicated, zone status of each reservoir is checked and if "hold" is applicable, the reservoir release is reduced to minimum outflow regardless of release established by other routines. The Missouri and Mississippi River "holds" are entered as input data for each time period. Format of code: A - 13.

No. 14 - End of Computation.—Read in reservoir inflows for the current time period. Compute new reservoir storages. Store reservoir releases and storages for the time period.

APPENDIX C.—RELATION BETWEEN SUBAREAS PART I AND REACHES PART II AND III

TABLE 4.–RELATION BETWEEN SUBAREAS PART I AND REACHES PARTS II AND III

Reach Code Number Parts II and III (1)	River (2)	From To (3)	Subareas - Part I: Code No. (4)	Subareas - Part I: Computation No. (5)
1	Smoky Hill	Kanopolis Dam to Lindsborg	12 13	Kanopolis Inflow[a] SH-11
2	Smoky Hill	Lindsborg to Mentor	14	SH-12
3	Saline	Below Wilson Dam	6 7	Wilson Inflow[a] WOL-C
4	Saline	Below Wilson Dam to Lincoln	8	SA-6
5	Saline	Lincoln to Tescott	9	SA-7
6	Saline Smoky Hill	Below Tescott and Mentor to New Cambria	10	MUL-C
7	Smoky Hill	Above New Cambria	15	SH-13
8	South Fork Solomon	Webster Dam to Alton	16 17 18	Webster Inflow[a] SFS-5 SFS-6
9	South Fork Solomon	Alton to Osborne	19	SFS-7
10	North Fork Solomon	Kirwin Dam to Downs	20 21 22 23	Kirwin Inflow[a] NFS-5 NFS-6 NFS-7
11	Solomon	Downs and Osborne to Beloit	24 25 26	NFS-8 SFS-8 SO-1
12	Solomon	Beloit to Minneapolis	27 28	SO-2 SO-3
13	Solomon	Minneapolis to Niles	29 30	SO-4 SO-5
14	Smoky Hill	Niles and New Cambria to Abilene	31	SH-14
15	Smoky Hill	Abilene to Enterprise	32	SH-15
16	Republican	Harlan County Dam to Guide Rock	33 34 35 36	Harlan Co. Inflow[a] R-12 R-13 R-14
17	Republican	Guide Rock to Hardy	37	R-15
18	Republican	Hardy to Scandia	38 39	Lovewell Inflow[a] R-17
19	Republican	Scandia to Concordia	40	R-18
20	Republican	Concordia to Clay Center	41 42 43	R-19 R-20 R-21

TABLE 4.—*CONTINUED*

Reach Code Number Parts II and III (1)	River (2)	From To (3)	Subareas - Part I: Code No. (4)	Subareas - Part I: Computation No. (5)
21	Republican	Clay Center to Milford	44	R-22
22	Republican and Smoky Hill	Milford and Enterprise to Junction City	45 46 47	CHAC LYO-C SH-16
23	Kansas	Below Junction City	48	CLK-C
24	Kansas	Junction City to Manhattan	49	KRL-la
25	Kansas	Manhattan to Wamego	50 51	Tuttle Cr. Inflow[a] KRL-lb
26	Kansas	Below Wamego	52 53	VER-C1 VER-C2
27	Kansas	Below Wamego to Topeka	54 55 56	MIL-C CRC-C KRL-2
28	Kansas	Below Topeka	57	SOL-C
29	Kansas	Topeka to Lecompton	58 59	KRL-3 DEL-2
30	Kansas	Above Lecompton	60 61 62	DEL-la DEL-lb DEL-lc
31	Kansas	Below Lecompton	None	
32	Kansas	Lecompton to Midpoint	63 64 65 66 67	WAKR-1 WAKR-2 STRC-1 STRC-2 KRL-4a
33	Kansas	Midpoint to Bonner Springs	68	KRL-4b

[a] Use Inflow for Part I and Outflow for Part III.

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SOME LEGAL ASPECTS OF SEDIMENTATION

By C. E. Busby[1]

SYNOPSIS

Legal aspects of sedimentation, of interest to the engineer, geologist, and soil scientist, are presented. The paper concerns rights in land, as deposited ediment and rights to water, as containing or transporting sediment. It also ontains the rights to be free from undue damage caused by artificial changes n the movement and effects of water, involving sedimentation as a process.

INTRODUCTION

Outline of Problem.—This paper deals with some of the legal aspects of edimentation, of interest primarily to the engineer, geologist, and soil cientist. It concerns rights in land, as deposited sediment, and rights to vater, as containing or transporting sediment; and rights to be free from undue amage caused by artificial changes in the movement and effects of water, nvolving sedimentation as a process.

The process of sedimentation—weathering, erosion, transportation, depoition and consolidation—will be followed in this statement, insofar as the ositive law will permit. At each point the persuasive rules of local custom nd practice will be anticipated or referred to when summarizing the coercive ules of law. However, it should be recognized that many of the rules of roperty and damage rights are really legal policy guides governing relationnips between people, as individuals and groups, and their resources, and

Note.—Discussion open until December 1, 1961. To extend the closing date one onth, a written request must be filed with the Executive Secretary, ASCE. This paper part of the copyrighted Journal of the Hydraulics Division, Proceedings of the Amerin Society of Civil Engineers, Vol. 87, No. HY 4, July, 1961.

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that the rules of custom and rules of law are interwoven in the fabric of social controls and agreements (1).[2]

Each such rule of policy has built into it features (a) of objective, purpose, and use that serve to channel human energy toward social goals, and (b) of definitions, conditions and limitations that serve to set depth, width, slope, and velocity guidelines for control of that human energy.

The implementation of these policy rules in terms of administrative organizations and programs, through grants of governmental powers, involving responsibilities in the field of sedimentation, is so large a subject that only certain aspects of regulation can be dealt with in this paper. Again, however, these delegated powers are merely broad rules governing the relations among people and their resources in a governmental setting. They add up to the force that moves human energy through the policy channel.

Sediment yield is here restricted largely to erosion and its immediate on-site consequences (as a process) and does not include weathering, as such. Similarly, the vast field of water law is treated here only in skeleton form to bring out certain of its sedimentation aspects, recognizing, however, that every major quantity depletion, replenishment, and storage of water may have sedimentation and other consequences by altering the natural process at various points.

Key questions are these:

1. What rights in and to sediments, as land (property), are recognized in law as arising out of natural and artificial changes in the movement of water?
2. What rights to legal damages are recognized in law as arising out of artificial changes in the movement or effects of water and wind, with special reference to sedimentation?
3. What powers of government are recognized in law as necessary to regulate land and water use in order to prevent undue damage by sedimentation to resources and to the health, safety and welfare of the community?

Answers to these questions usually revolve around concepts of public necessity and interest that change with the changing times. It is important, then, that scientists recognize the changing character of the physical and social processes (2).

Legal Concepts of Water, Air and Land.—It is commonly recognized that one must have possession (of property) or the right to possession as the basis for rights in land and water. The nature of what is meant by possession differs somewhat for the two types of resources, separately and together. In some cases physical possession is impossible or impractical, and so the legal idea of constructive possession comes into play.

When negligence, nuisance, and compensation are involved, it is also recognized that an injured party must be able to trace and prove as a matter of cause and effect that damage initiated and brought about by the action of another is the cause of material and substantial injury to others or their resources.

Furthermore, people who bring about such damage are liable for all the material consequences of their unlawful acts, save those outlawed by the statute of limitations or by some other legal defense, such as contributory negligence, laches, or estoppel. Such consequences may thus extend the effect

[2] Numerals in parentheses refer to corresponding items in the Appendix.—Cou Citations and References.

of sedimentation damage to other types of damage, such as some forms of pollution, flooding, and waterlogging. Erosion damage is here considered as a part of the sedimentation process.

In American jurisprudence, one cannot own the water as it runs in a stream or moves in the air, for one cannot legally possess it in these natural states. This has given rise to legal concepts as old as Roman Law; that these moving waters are the property of no one (res nullius) or of all people (res communes). All one can own of such moving resources is the right to take possession of and use them under certain conditions; the right to the fruits of such use; and the right to transfer these rights to others. There is also the legal concept of public property (res publici) that emphasizes public control of the resource within a given jurisdiction, such as a state (3).

Thus, most land and water rights are recognized as real property, and, in the case of water, until it is taken into possession, when it is considered personal property. But what constitutes personal property in water varies somewhat among the several jurisdictions; one can only own a "usufructory" right to water as it flows in a stream.

Presumably, these concepts apply to the sediment in suspension or otherwise, but this has not been well settled by the court decisions because definite values have not yet been attached to it. However, it should be noted that some streams carry certain types of sediments in suspension that are valuable as mineral and organic fertilizer. The Nile and Mississippi are good examples.

The Erosion and Sediment Process Varies Geographically.—In considering these aspects of our natural resources, it may be well to keep in mind that the Midwest, Southeast, and Southwest vary greatly as to rock and soil material susceptible of erosion, transport, and deposition. And, also, that these geographic sections differ markedly as to the rate at which the sedimentation process is going on, being measured, and recorded.

The difference between the natural rate and any artificial rate produced by man is significant in fixing legal liability. But this may be a difficult matter of proof calling for the best scientific methods and techniques, both in engineering and in the law of evidence because the settlement of actions at law calls for reasonably ascertainable facts as to damage and some reasonable finality as to a settled accounting. Otherwise, fairness is not really possible and multiplicity of suits is to be anticipated.

The facts of sediment damage are presently more evident and measurable in larger stream valleys, such as the Mississippi, Sacramento-San Joaquin, Rio Grande, Colorado, and Red. Here the requirements of legal proof can more easily be satisfied than in small stream valleys owing to better and more reliable records. The public interest in these large valleys is greater, too, for example, in terms of navigation, flood control and drainage.

But the units of government and their agencies, owning and controlling land and water, cannot be sued for damages without their consent. Even if they do consent, their legal liabilities may be limited where the stream is a public or navigable one. Today the concept of what is a public stream is changing, as the needs of the public change. And so is the public interest in small streams and watersheds.

One of the most difficult problems in the field of sedimentation law is how to arrive at a final accounting of legal damage in the face of a physical process that changes over a span of years, during which natural processes of control become established.

The Land Pattern Affects the Process and the Legal Consequences.—The land, from which sediments arise, occurs in natural patterns of rock and soi that reflect the geology, climate, vegetation, and topography of the earth's crust, now and in times past. But superimposed on this natural pattern is a cultural one that affects sediment production through land and water use. In like manner, the land that sediments produce also occurs in natural patterns and various cultural patterns are gradually superimposed on these.

In the civil law jurisdictions (Louisiana, Puerto Rico, parts of the South-west and Michigan, and Quebec), the private land pattern often consists o long, narrow strips that extend back from the water's edge for various distances. Often these individual strips are fenced to permit exclusive use. This pattern reflects the riparian right of access to the water. But it tends to limi the opportunities for diversion and use of water (and included sediments) o the land owing to watershed and fence boundaries; it also tends to lead t subdivision into ever-narrower strips by reason of inheritance laws. At Taos N. Mex., some strips are only a few feet wide and half a mile long.

As a major example, the lower half of Louisiana is one vast sedimentatio deposit (having its source in upstream states), subject to civil law rules c water use and management. Here is a vast area in which sediment deposit have created fertile lands, with the coarse red sands along the rivers an bayous, and the fine dark clays in low swales away from the channels. Bu along the south seacoast, the soils formed from such deposits are subsidin under salt water due to superimposed new sediments. Extensive efforts ar being made there to prevent encroachment of contaminating salt water

In the common law jurisdictions, the shape of land ownership tracts fo settlements of the original states is often determined by metes and bound (using stream thread or bank, as bounds, in some cases), or by sections an townships that ignore drainage lines for younger settlements of the newer admitted states. The Indian land pattern seems to have a rough resemblanc to both, as at Taos. There are certain areas in the Mormon communities c the West that resemble the urban, wheel-like pattern of some European settle-ments.

In addition, there are public geographic boundaries of land areas, such a counties, municipalities, states, and the nation; and such as national forests public parks, wildlife refuges, and so forth. These are either ownershi boundaries or boundaries wherein governmental (police) powers may b exercised over natural resources and people. Any jurisdiction may declar public ownership of water within its boundaries and within its authority. B this has reference mainly to sovereign control of development and use, rathe than to actual possession of the resource.

The Water Pattern Affects the Process and the Legal Consequences.—Th natural pattern of water occurrence over these natural and cultural land area depends on slope, soil, bedrock, gullies, and stream channels. But superim posed on this is the invisible cultural pattern of water supply and rights use, as defined and classified in law, such as diffused surface waters, vagra flood waters, and defined stream and lake waters (watercourses). The latt are further defined and classified as either navigable or non-navigable, just public for various reasons reflecting the current use needs of the peopl The original common law concept of navigable waters was based on the e and flow of the tide, such as to float a sailing ship.

There is some difference as to how vagrant flood waters are presently defined from state to state, and the same is true for navigable waters. The federal test of navigability is especially important in determining title to stream beds and to sediments that have come to rest in these beds and on the flood plains.

Throughout the English-speaking world, there is usually a distinction made between the positive or existing law of record and the law as it ought to be in the minds of scholars. This division will be observed here with special reference to the need to bring law and science closer together in terms of reality as process, such as the water cycle, the sedimentation process, the human rather than supernatural source of rights, and the interests they are designed to secure, and so forth. It is interesting to note in passing that much of the law of sedimentation is more nearly in line with the process character of reality than most other aspects of resource law. This is important to resource management.

These factors of natural and cultural patterns of resource ownership and use are outlined here because they affect engineering surveys, appraisals, plans, and programs, as well as matters of law and administration. And because rights are often a matter of means of measurement and proof, as well as of definition and classification. The facts usually make the case.

Sources of the Law.—There are two major sources of the positive law of sedimentation, civil and common, as modified by statute, compact, treaty, and constitution. The first, originating in Rome, was codified by Justinian. But the development of the law was somewhat different in respect to rights to water and sediments in France under the Code Napoleon and in Spain under Las Sieta Partidas, etc., and their respective possessions (Louisiana, Puerto Rico and Quebec).

In both areas local customs had a marked influence on the law as it evolved. The French version tends to emphasize non-consumptive uses of water, whereas the Spanish version tends to emphasize consumptive uses, though the latter was affected by the former after the Code Napoleon was promulgated. Differences in climate may have had an influence, too.

In these civil law areas, the legal method of defining water supplies or sources has been rather uniformly one of dividing up of the water cycle into many parts, and expressing each division in terms of land catchment areas; then the private and public rights and responsibilities for each such division and area were assigned, as well as for the cultural land pattern superimposed on it. This is not fully in line with scientific fact and, therefore, partially unsound because it tends to prevent conservation and wise use of the supply in certain areas and unified administration of the law.

The process character of reality is the vital factor that is overlooked. The old legal definitions seem to reflect efforts to achieve greater certainty and stability but in the face of a process that is constantly in a state of change. However, these matters could be adjusted quite readily by applying more modern legislative methods of definition and by improving certain methods of judicial interpretation.

Louisiana, Puerto Rico, and Quebec emphasize water rights in water-courses mainly for those who own lands touching stream or lake banks. Nearly all of our eastern states follow these general rules, too. Some of these states permit diversion and limited uses on non-riparian lands, but do not usually

permit seasonal storage of water by private persons, unless especially authorized by statute in the public interest.

In the western states, save for some remnants of the riparian rights of use, the prior appropriation law has been adopted as of necessity, to make possible needed depleting uses on suitable lands in a dry climate, together with the structural means of conserving water by storage. These uses and controls tend to deplete the natural flow to some extent in time and place, and, therefore, can appreciably affect sediment transport and deposition and also water quality.

Most of our states still follow some form of the English common law of diffused surface waters (precipitation runoff) which is not at any particular moment part of a stream. "Surface waters" is the term used in most official law reports. This is based on the theory that one who has title to the soil also owns all waters on and under it, as well as the space above and the minerals below.

This theory is not fully in line with scientific fact either, for many moving resources, such as water, oil and gas, air, and wildlife. For that reason it is partially unsound. However, the rates of ground water movement are much slower than for stream flow, so the legal concept of possession should be adjusted accordingly. This old "cujus est solum" theory affects all water supplies because it affects every land ownership tract.

The various civil and common law concepts, including definitions of water supplies, have found their way into our State and United States Constitutions through various legal processes. These important provisions affect rights in water and land and rights to damage arising from sedimentation. The supremacy, due process, and commerce clauses are the most important. The due process clause is especially important as to sediment damage and its consequences. But its application is affected or limited by the commerce clause.

These constitutional provisions have been implemented by statutes concerning navigation, flood control, and drainage; hydroelectric power, recreation, and fishing; pollution, salt water contamination, and sediment control and water allocation and use. They are too numerous to summarize here even though they could be included under the subject heading. Only special ones will be mentioned in this paper.

Suffice it to say, the broad trend of the law is away from older unscientifi concepts and judicial administration toward more scientific concepts an executive administration. In this transition, the engineering profession i playing a very important role by developing and applying basic scientific dat within the broad framework of legal administrative processes and standard of fair play.

Much remains to be done, however, in order to bring definitions of wate supplies, sedimentation and saline processes, and theories of property owner ship and use in line with the process character of reality. One can see thes trends and needs by examining, progressively, some of the broad rules of la and how they have changed over the years and over the nation.

In this summary statement it will be necessary to use much secondar material in presenting the subject and, at the same time, to make som assumptions regarding rights in and liabilities applied to sediments and sedi mentation as a process, that seem to follow from legal rules applied to wate and land. But it is hoped that eventually the subject can be presented in greate

detail, so as to cull from the body of sedimentation law, as it is developing over the nation, the full story of this challenging field of social engineering.

RIGHTS IN LAND AND DIFFUSED SURFACE WATERS

Definitions of Supplies and Interests.—Precipitation accumulated on or running off the land is known as diffused surface waters in law, though the courts usually use the term "surface waters." Presumably, these have no defined channels in a legal sense, thus the term "diffused." Engineers know that water is usually in some sort of "channel" as soon as it starts to move over the land.

A channel ("watercourse") in law is one having bed and banks where the water flows when it normally does flow. Though this can be intermittent, there must be a "dependable" source of supply. Actually, the channel is merely a physical means of draining surface and ground waters. And the channel ought to be distinguished in law from the water supply source, as such, rather than confused with it. The channel affects the external properties of water but it is the internal properties of water that are most important (4).

Thus, the courts have divided the water cycle into diffused and defined surface waters. And then made an even more unscientific division, separating these waters from ground waters ("percolating" and "in defined subterranean watercourses and reasonably ascertained basins"). However, this was done when judges, as well as people generally, didn't understand the water cycle. And when there were few good precipitation and flow records. A distinguished judge has said that it is these men of authority who govern and not our legislatures; that they are the ones who formulate public policies and our basic law (5). If so, they should become better acquainted with applicable engineering principles.

Because land is commonly referred to in law as the absolute property of the owner, so, too, diffused surface waters are sometimes referred to as the absolute property of the landowner, in which they occur, for the title arises from ownership of the land (6).

Exceptions to that are the statutory, constitutional, treaty, and compact limitations in several western states, to the effect that all waters or all surface or running (stream) waters are the property of the state (or people of the state), or the respective states which are parties to the compact or treaty (7).

Relative rights to the use of these diffused surface waters have not been settled in the laws of our states, though the type and size of dams and purpose of reservoirs built to conserve these waters are regulated by statute in some western states from the standpoint of health and safety and to a limited extent from the standpoint of storage uses. More of this type of regulation may be anticipated in the more critical supply areas.

There seems to be a tendency for some western courts to give the stream user a legal interest in diffused surface waters and in underground waters originating above and within the watershed to protect the established stream appropriator's interest. This is especially true in Colorado. But how to make this fully effective is a difficult administrative problem when the ground water aquifer resembles a broad porous blanket.

In reviewing this aspect of the law, as applied to sediments carried and deposited by these waters, it may be well to think of diffused surface waters

as (a) occurring in depressions or (b) draining off the land through rills rivulets and small swales, and (c) combinations of these physical features Rights to possession of the land also provide the primary legal basis fo rights in and to these waters and sediments, as presently defined, except a indicated previously.

Thus, it would appear that sediments that have been deposited in thes depressions and related physical features become the property of the land owner by reason of his possessory interest. By the same token, this could b true in large part for sediment deposits in downstream valleys below, an resulting from this type of runoff, even though these deposits may be in one' "possession" during one year and in transit and out of possession durin another year. There seems to be little case law that specifically spells ou these types of rules, as yet, for diffused surface waters and their "host" lanc The law of streams is more definite in this respect.

In many densely populated areas of the world, fertile soil is so valuab that where accumulated as sediment on lower slopes, it is picked up in basket and carts and carried upwards and then redeposited by hand on higher slope to improve the eroded soil. In these circumstances, the "sediment-soil" ha great value as property. Presumably, its possession affords a legal basi necessary to redistribute, conserve, and use it on the land. Of course, onc it has left one's land and been deposited on that of another, the right of pos session is probably lost. This aspect seems to have been recognized in Roma Law (8).

Common Enemy Rule.—From these "guideposts" of what the positive la is or might be in terms of definitions of supplies and interests under cond tions of human occupation and need for land, one can now examine the existi rules on rights to use and change the movement of diffused surface wate and their included sediments over the land surface. In other words, the rules define the rights that apply to the interests that supplies can serv

The so-called common enemy rule prevalent in Arizona, Arkansas, Co necticut, District of Columbia, Indiana, Maine, Massachusetts, Mississip Missouri, Montana, Nebraska, New Jersey, New York, New Mexico, Nor Dakota, Oklahoma, Rhode Island, South Carolina, Virginia, Washington, We Virginia, and Wisconsin (9) holds that the landowner (by reasons of his po sessory interest) can treat these runoff waters as a common enemy and with them as he pleases in order to protect his land and its use. Apparent New Zealand and most of the Canadian provinces, except Quebec, have t common enemy rule, too. Iowa, Kansas, Pennsylvania, and Texas adopted th rule initially and then changed to the civil law rule. This was said to be t common law rule and was adopted by some states as rule of decision for t court when the states were admitted to the Union or thereafter (10).

Such landowners can throw these waters back on higher or adjacent lar and should this result in (water and sediment) damage to these lands, no le remedy is afforded the injured party. The courts refer to this as dama without legal injury (damnum absque injuria). But the public policy used justify such a rule varies from one jurisdiction to another.

This general rule has been modified in some states by exceptions, su that one landowner may not needlessly, or carelessly, or maliciously injury to other land by changing the movement of "his" diffused surface wate That is, any such injury that substantially impairs the value of the land its use creates a legal liability. The injury might take the form of harm erosion, sediment accumulation, waterlogging, or pollution (11).

In other words, the so-called absolute property right in land is not really absolute after all. It is qualified by exceptions in several states in the interest of the rights or needs of neighbors. These limited exceptions find fuller expression in the rule of reasonable use (discussed later) that has permeated our whole body of land and water law (that is, the rule of reason). This rule could afford the opportunity for incorporating more science into the law because sound reasoning calls for consideration of relevant scientific fact, method and technique. Judges could, with confidence, open up the rules of evidence to let in more scientific facts and principles, especially when supported by local custom and practice.

Civil Law Rule.—In several other states and territories, the common enemy rule has been replaced by the civil law rule, or only the civil law rule was adopted there. This holds that the upper or higher landowner has an easement over the lower landowner for the natural flow of "his" diffused surface waters (12). Alabama, California, Colorado, Georgia, Illinois, Iowa, Kansas, Kentucky, Louisiana, Maryland, Michigan, Nevada, North Carolina, Ohio, Pennsylvania, South Dakota, Tennessee, and Texas. Quebec and Puerto Rico apparently have this rule, too, and possiblu Australia, India, and Scotland. Presumably, the states of Mexico also have it. In a state of nature, this flow obviously would include some sediment. But when the land is improved, whether in rural or urban areas, the volume of runoff water and sediment may be greatly increased, especially under poor watershed management.

The problems arising out of improving land for various uses have led to the adoption of exceptions (really reasonable use rule) that the upper landowner may not unduly collect, concentrate, and discharge diffused surface waters on the lower land in increased velocity and volume, so as to do substantial injury to the lower lands. These exceptions have many ramifications.

Most state courts do not yet seem to recognize a right in the lower owner to have diffused surface waters (and sediment) come down to him, so he could use them beneficially, except for "dicta" in such states as Colorado and Utah, tending to give lower stream users some use rights in these waters. Presumably, the reasoning rests on the assumption that these waters do not have a defined and regular source or course.

But such is not really true for they do have a recurring source in the water cycle. And their courses often are relatively dependable, though not deeply chisled out of the soil and bedrock. Though subject to overflow and spreading over the land in flat areas and under conditions of excessive runoff, the flow characteristics differ mainly in degree from that of streams that overflow their channels. In England it appears that the lower stream user has no tangible use claim to diffused waters in excess of what ordinarily does come down to him (13).

Thus, we see that from an old common law rule that permitted almost unlimited obstruction of these waters and sediment (favoring the individual interest of the lower owner to protect his land) or an old civil law rule that permitted little or no obstruction (favoring the individual interest of the upper owner to get rid of his excess waters), we find the law changing toward a rule of reason that tends to balance the relative interests of upper, lower, and adjacent landowners as to damage resulting from harmful runoff. Thus, their relative rights to be free from undue water damage caused by acts of man seems to have become more nearly correlative or reciprocal, as the land becomes more fully occupied and used. But the basic problem of the courts

in defining these rights and the interests they are intended to serve, is th scientific one of how to measure, evaluate, and predict runoff and damage And this is the task for engineers.

This change in the law merely reflects emphasis on the modern needs c society in relation to the needs of the individual; that is, the common need c the community to share the whole resource. However, the law has had littl real development in regard to rights to use diffused surface waters. This ha been left largely to local custom and practice of private landowners and loca agencies, such as soil conservation districts, flood control districts, an drainage districts.

It has been brought out in various publications that the courts have mad efforts to adjust the common enemy and civil law rules, working from the tw extremes indicated above toward the reasonable use rule. One aspect ha dealt with the capture and use of these waters by artificial means (14

A second aspect has dealt with the disposal of excess diffused surfac waters by artificial means. These cases, (in Arizona, Arkansas, Connecticu District of Columbia, Indiana, Massachusetts, Mississippi, Missouri, Nebras ka, New Jersey, New Mexico, New York, North Dakota, Oklahoma, Rhod Island, South Carolina, Virginia, Washington, West Virginia, and Wisconsin discuss types of improvements that may be used, the consideration of goo husbandry and improvement of the soil (15), and the volume of discharge in drainageways (16).

The third aspect has dealt with obstructions to the natural flow. And fourth concerns the so-called reasonable care fule. These seem merely to steps toward adoption of the reasonable use rule but involve a wide varie of policy concepts that are somewhat conflicting. What is really needed is new concept of using the resources according to capability and treating according to need.

The Reasonable Use Rule.—The landowner, under modern conditions, is n unqualifiedly privileged to deal with diffused surface waters as he please Nor is he absolutely prohibited from interfering with the natural flow to t injury of others. He is legally privileged to make a reasonable use of his lan even though the flow of water is altered thereby and causes some harm others. He incurs legal liability only when his interference with the flow water is really harmful and clearly unreasonable in the circumstances. It like a lack of respect for others and for the land.

This rule was first applied in New Hampshire. Apparently, only a f states have the rule as yet, such as New Hampshire and Minnesota (1

The rule, in effect, is this: a landowner may use his own land as he pleas provided he does not unreasonably interfere with like rights of others. Reaso ableness and unreasonableness are questions of fact. It is the amount of ha caused in relation to the value of the improvements made; that is, the natu of improvements made or ought to be made, the nature and extent of the inte ference, the foreseeability of harm of one making the improvements changes in flow, the purpose or motive with which he acts, and other relev matters. Actually, these factors do not get at the real issue of capabilit and needs of the resource (18).

Every interference with the drainage of diffused surface waters, actua harmful to the land of another and not made in the exercise of reasonable of one's own land, may be considered unreasonable. The rule does not down any specific rights or privileges but leaves the policy of the law depend on the facts of each case and the principles of fairness.

This makes a lot of sense if scientific facts are used. But it also provides limited guidance for landowners. Here is where local custom and practice become so important, for if they are scientifically established and then accepted, the courts have a sound basis for good decisions.

In some states it is recognized that the rule needs to be applied differently in agricultural areas, as compared with urban areas (19). The common law, then, has been modified in most states, even where the other two rules are applied, to this effect: a person may so use his own property as not unreasonably or unnecessarily to do material injury to his neighbor (20). The courts must reconcile the benefits with the harm caused.

Sometimes the terms "negligence," "trespass," and "nuisance" are applied to harmful changes in the natural flow of diffused surface waters, but without much consistency and precision. These remedies, in several different legal forms, were given a great deal of consideration to Roman Law (21). They could be used more fully today if the forms of action for redress of harm were built more around local custom and conservation experience, now receiving widespread consideration in local soil and water conservation districts. In other words, judges could lift up modern local experiences and incorporate these into local rules of law with real confidence that scientific facts and judgments had been used in their development. Such experiences are based on capability and need concepts of public resource policy.

Pollution Damage by Sediment.—Numerous court decisions deal with the liabilities of upper landowners for damage to land and water supplies below, resulting from sediment and other impurities introduced into diffused surface waters as a result of land use practices. These cases may or may not speak of sediments, as such, but usually speak in terms of soil, rock, mud, sludge, harmful chemical or biological substances, including atomic or radioactive wastes.

It has been held that the upper landowner is not liable for damage to lower lands caused by diffused surface waters carrying soil and rock when they constitute part of the "natural formation of the land." But he is liable for resulting damage if he places other soil and rock where the natural drainage of such water will carry it to lower tracts or where it interferes with the normal drainage (22).

It has also been held that the riparian owner is not liable for impurities or pollution that find their way into a stream from the natural wash or drainage of his land within the upper watershed (23). But it is unreasonable use of one's riparian land to dig up swamp and (negligently) or to sluice large quantities of loose mud into a stream and thereby cause (pollution) damage to a lower property owner.

Many cases similar to these have resulted from mining and milling operations that sluiced sediment-type debris into streams, so as to pollute the water and land by acid and other harmful substances, cause overflow of the channel and flood damage, and reduce navigable channel capacity (24).

Hydraulic gold mining in California, where it leads to the reduction in navigable channel capacity due to sedimentation, came to full regulation after 1884, with only limited operations permitted (25). A situation something like this, involving iron and steel manufacturing, has arisen on the Calumet River in Illinois, that now empties out of Lake Michigan and connects with the Mississippi River system (26).

Here there were new legal questions to be answered in the light of changed needs of the public interest today, such as: What is a legal obstruction to navi-

gation under the Rivers and Harbors Act? What is refuse matter? Where is the dividing line between sediment material that constitutes such an obstruction in fact and in law and that does not?

One similarity in kind in these two situations lies in the fact that the debris was of grade size that could be carried in suspension by flowing water and later deposited with change in velocity in a navigable river channel. But beyond this the two problems differ in degree.

In the mining instance, the debris resulted from the washing of recent alluvial stream gravels and from sluicing the suspended silt and fine sand into the river after the gold particles had been removed. While in the manufacturing instance, the debris consisted of flue dust, iron oxide, slag, and calcine lime. This resulted from iron and steel making and from sluicing the very finely flocculated material either directly into the river or into the sewers and thence into the river in which it deposited under relatively low flow velocity (27).

The grade size of the latter was below 325 mesh screen. Expressed otherwise, 90% fell below $62\frac{1}{2}$ microns and 80% fell below 31 microns. A micron is 1/1000 of a millimeter. In contrast, the largest molecule in solution is said to be about 0.001 microns. The companies used settling basins to remove most of the debris but enough passed on so as to create shoaling in the close-by channel. Some of the material may have been colloidal iron oxide (though the court record does not say so) because much of the suspended load could not be seen under the microscope, as indicated by expert testimony (28).

In the mining situation, there was specific legislation so stringent in regulation as virtually to force abandonment of the industry, except in rare instances, under prevailing prices of gold. In effect, this was also a drastic regulation of land use where the dredging takes place. In the manufacturing situation there was only general and relatively old legislation that was subjected to rather strained legal interpretation by the Supreme Court largely in terms of certain definitions. This is shown by the 5-4 decision, detailed technical arguments of the dissenting judges, and sharply contrasting concept of public policy as between the lower and higher courts. Here is an example of how a basic policy is changed merely by changing certain definitions (29).

The Republic Steel case re-interprets the broad general policy provision of Section 10 of the Rivers and Harbors Act of 1899, as amended, that says "That the creation of any obstruction not affirmatively authorized by Congress to the navigable capacity of any of the waters of the United States is hereby prohibited."[3]

The words, "any obstruction," are, by this decision, thereby no longer limited to works of improvement, as argued by the dissenting justices and the Circuit Court (referring to prior cases), but now also include industrial wastes of a very fine size when they reduce navigable capacity. The words "navigable capacity," are already interpreted broadly enough in other court decisions to reach almost any stream capable of being made navigable by works of improvement.

It is of interest here that previous opinions of the Attorney General concerning the Act (Op. 21, 305, 1896, and 594, 595, 1897) had attempted many years ago to draw a line between suspended materials and materials in solution by defining an obstruction to include other than works of improvement. The present decision seems to adopt the substance of those opinions (30

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[3] The italics are the writer's.

Thus, it is of special significance to engineers that the interpretation of Section 10 of the Act now includes industrial solids in suspension but not in solution. And the interpretation also limits the exception clause, too, Section 13, to apply only to refuse flowing from sewers in a liquid state as sewage; that is, organic wastes. The court implies that organic wastes react chemically on discharge into a stream, so as not to remain permanently as an obstruction in the form of a shoal deposit.

In subsequent situations, any industrial wastes sluiced into a navigable water of the United States or indirectly by way of tributaries to a navigable water body, whether direct or through municipal sewers, so as adversely to obstruct its navigable capacity, may be subject to the control of the Corps of Engineers, and to injunctive relief on failure to comply with Corps regulations. It could be a short step beyond this to apply these principles to other wastes and to silt from agricultural watersheds if the injury to navigable capacity could be traced with reasonable certainty to the source, as a sequence of related physical events.

Thus, for example, the Republic Steel case may have indirectly overruled the Brazoria District case in which silt resulting from faulty drainage ditch outlets and discharged into and adversely affecting a navigable channel were previously held not subject to injunction. In that case the line defining an obstruction was drawn on the basis of a work of improvement, a structure, not on the basis of sediment in suspension that later became a deposit. At the same time, the Steel case tends to lend weight to the Sand and Gravel Case (31). It would seem worthwhile to relate these sediment problems, involving regulation, to those that follow, involving compensation under the 5th Amendment to the United States Constitution.

RIGHTS IN LAND AND DEFINED SURFACE WATERS

Definitions of Supplies and Interests.—Waters accumulated in or flowing through channels with well defined beds and banks are known as the waters of "watercourses." The courts tend to treat the channels and the waters in them as one and the same thing in the use of language but separates them, usually, as to ownership and jurisdiction. They should be treated in law as well as in science as separate and distinct, though closely related in terms of the external energy properties of water (32).

The law of watercourses, as it relates to sedimentation, involves riparian rights to land (involving deposited sediment and access to water), and to the flow and use of water (including storage, diversion, and disposal of waters containing sediments in suspension and solution). It involves rights to be free from undue damage caused by water and land use (involving injury to property done by overflow, erosion sediment deposited in channels and on the land, as well as other harmful consequences of interference with the flow of water), such as waterlogging and salinization of lands and reduction in navigable capacity.

Few court decisions speak directly of these matters, as they relate to benefits from sediments carried in or deposited by these waters. They speak mainly of legal damages and compensation from interference with the flow impairment of the quality of the water. Eventually, the law may recognize rights to sediment in transit and, also, rights to be free from undue damage caused by sediment wastes placed in streams.

Streams and rivers, as running waters, are not the absolute property of anyone, as previously noted. And one can acquire the rights referred to above only by acquiring riparian land, or by making beneficial use of the waters, or by suffering injury or the threat of injury to property through the harmful acts of others, or by using public waters and channels as part of the public.

The conditions under which these rights are acquired and lost are important to engineers (and their clients) because engineers are called on to render services in measuring and appraising land and water resources and in evaluating property damage resulting from their control and use. Engineers need to have some familiarity with the legal rules of property ownership and boundaries and use values at the time surveys and appraisals are made (33).

Rights to Riparian Land as Deposited Sediment.—Rights to sediment deposited in stream channels or on flood plains may be gained or lost by changes in the position of the channel itself, especially its "thread," due to the action of water and sediment. The right of access to the channel and the right to the flow and use of water in it is the primary key to defining property boundaries and to the apportionment of the damage to land caused by erosion, overflow, and sediment. But there are also rights to be free from damage to property that has no frontage on the stream, such as railroads, bridges, and levees. Sediments affect the position of the channel in the flood plain by changing channel capacity and the topography of the surrounding flood plain (34).

Rights to deposited sediment depend on the law of the particular state in which they occur, unless the land is owned by or acquired from the United States. The rules vary somewhat from state to state, centering around the definition of navigable (or public) and non-navigable watercourses. The federal rule defining navigability is controlling as to ownership of bed and banks of the channel where title stems from the United States (35).

Riparian Rights Gained or Lost by Accretion.—Accretion, as a process i[s] defined in law as the slow, imperceptible addition of alluvial deposits (sediments) on the land margin or in the channel of a watercourse. When a strea[m] or river recedes below the low watermark, the exposed deposits are als[o] recognized as accretions. Apparently, the word "imperceptible" means to b[e] unaware of sudden change (36).

The change in the land must be permanent, however. The rule does n[ot] apply to sediment alternately above the under water, so long as the wate[r] substantially retains its old boundaries. This right is one of the bundle [of] riparian rights based on access to the stream which is the main guide [to] where the property lines are as the water recedes (37).

In the West, this right to accretions remains (in spite of the adoption of th[e] prior appropriation law), so long as the stream has not been diverted by a[n] appropriator. But the riparian owner has no vested right to have chann[el] conditions remain the same, such that accretions will continue to be forme[d] there in the future (38).

Under the common law of accretions, the water boundary shifts with th[e] stream (really the channel formed by it) and is not fixed. Thus, the later[al] lines extending from the banks toward the thread of the stream change wi[th] the channel thread, so that they are always at right angles to the thread [in] any permanent position of the channel (to preserve the right of access). Th[e] riparian owner may lose or gain land in this process. But the common la[w] may be changed by statute or words of an express grant, such as set forth

a deed. So it is important to anticipate these rules of the common law and of exceptions to it when surveys are made (39).

When accretions are added to an island on one side of a channel, and to the mainland on the opposite side, and then by a change in the river they are brought together as one continuous body, the physical union of the two tracks does not make the island a legal accretion to the mainland (40).

An island "rising" in a river and unconnected with the bank belongs to the owner of the bed at that place. This owner may be a private person or the State. The general rule is that the State owns the bed of a navigable watercourse, unless that State permits the adjacent riparian owner to own the bed, subject to the navigation servitude, as in Illinois and Mississippi (41).

Riparian Rights Gained or Lost by Avulsion.—Avulsion is defined in law as the sudden, natural change in the stream channel, so that the water flows elsewhere than in its previous course. Usually, this takes place during a major flood when the flow breaks through the natural dikes formed by sediments along the previous channel and flows into lower alluvial areas. This may produce degradation in one reach of a stream and aggradation in another reach, such that a new equilibrium is established (42).

The basic riparian right may be lost by this method when the thread of the stream is no longer the natural boundary. But the original owner may ditch the stream back to its former channel if he does not delay beyond a reasonable time. This time is a question of fact for a jury to consider. The owner has a right to take precautions by strengthening the banks against sudden changes brought on by freshets and washouts, if he can do so without trespassing on the land of another and without doing undue harm to another's land. Here, again, the rule varies somewhat from state to state (43).

Where the channel is changed suddenly and the stream abandons its former channel, the respective riparian owners are only entitled to the possession and ownership of the soil formerly under the waters to the thread of the stream, as it previously existed (44). If the change is gradual instead of sudden, as indicated above for accretion, the right is not lost because the accretion belongs to him with his won land and thus preserved his right of access.

Rights to be Free from Undue Damage Caused by Sediments.—The law in regard to damage caused by sediments has had much of its recent development in the federal courts and the U. S. Supreme Court, especially the U. S. Court of Claims (as to compensation under the 5th Amendment). But the rules in the early case law were developed mainly in connection with damage caused by water itself, such as overflow, and waterlogging of lands, resulting from dam construction. This has since been extended to certain other natural consequences of the building of structures and sluicing of debris into stream waters that change natural or existing water flow, sediment position and movement, and channel capacity. Eventually, it may include other related consequences, such as waste or loss of water supplies.

Sedimentation and other forms of related damage are now recognized in law, as in science, as a result of the more recent court decisions. But the full extent of the legal aspects of sedimentation, salt, and water damage effects of artificial obstructions (including debris itself) have not yet been fully developed by the courts. Ultimately, this may require improvements in methods for determining the nature and extent of damage caused by the erosion-sedimentation-salinization process, especially where there are other causes

involved, both related and unrelated to an obstruction placed in the stream channel. Damage consequences of change in the water table due to obstructions have only partially been settled in law.

Prior to the passage of the Tucker Act, there apparently was not an organized system whereby Congress could permit suits against the United States for damage resulting from dams and other improvements built by the government. With the passage of that Act, suits were permitted on the grounds of an implied contract to pay compensation for the taking of property under the 5th Amendment (due process clause). This principle was followed even though the property subject to damage was not specified and ascertained at the time of planning and construction (45).

But the argument was held for some time that the taking of property must be contemplated by government officials before an implied contract arose in law. If the legal complaint were based on tort (negligence, trespass, nuisance), there was no remedy at law under the Tucker Act because Congress had not authorized such remedy. Several cases were dismissed on these grounds. In fact, the proof of probable cause required by tort law could be very difficult when several miles of stream channel separate cause-and-effect areas, such as in the Southwest. And, also, where several industries are disposing of wastes into the same water body, as in the Chicago area.

So an implied contract was necessary to recover under the 5th Amendment, as then interpreted. But this method was complicated in the early cases by the lack of sound scientific methods, techniques, and information, particularly of an engineering nature, as to predictability of floods, rates of sedimentation, and related matters. In fact, this situation tended inadvertently at one time toward the legal conclusion that damage consequences were not contemplated, such as in the suit involving the Calaveras River, California (Sanguinette vs. United States) (46).

Taking of Property by Overflow.—However, in 1930, in the suit on Jones Creek, Ala., a tributary of the Tennessee River, the federal courts held that the Act of Congress authorizing the navigation dam and the findings made incident thereto were evidence enough that the taking was contemplated, even though the dam made only a slight increase in intermittent rises in streamflow and the overflows were only occasional, such that the full use of land for agricultural purposes was prevented for short periods. The dam caused increased legal burdens on the land in terms of more frequent overflows than normally occurred, and the resulting injury to use of the land called for compensation under the 5th Amendment (47).

The court organized its reasoning first on the basis of the riparian (property) right to the natural flow without burden or hindrance by artificial means. And, second, that no public easement for overflow can arise without government grant or compensation. The relationship here is between the private owner who is damaged, and the Government. But it should be noted that this uses the natural flow rule not the reasonable use rule (48).

The difference in the early cases is that in the Cress case the land had not been previously subject to overflow before the dam was built but was afterwards, whereas in the Jacobs case the land was subject to overflow before the dam was built, but the overflow was increased afterwards. The courts recognized the difference as one of degree, not in kind, but, nevertheless, fully compensible. Attempts to extend this "degree" theory of legal damage to sedimentation consequences have not been wholly acceptable, as yet, in some

other cases, as shown by the close decisions and dismissal of some suits on grounds that the causes were legally too remote.

Taking of Property by Overflow and Erosion.—From overflow alone, as in the foregoing cases, the courts have moved to the next step of considering resulting erosion damage. This arose after construction of the Winfield Dam on the Kanawha River, leading to both intermittent and permanent flooding of property and to erosion of the banks of the pool. The government could have purchased the land and necessary easements (for intermittent flooding) but chose not to do so.

The suit was brought under the Tucker Act as on an implied contract. The dam caused a rise in the river level that led to both the overflow and the erosion damage. The court treated these as related aspects of a continuous series of events not a single event. This very important case seems to have brought about a major shift in the basis of legal recovery away from implied contract and placed it squarely on a claim stemming from the 5th Amendment itself. Once established, this approach cannot help but be applied to other related aspects of the sedimentation process, so long as the damage can be fully ascertained and the cause of it proved with reasonable certainty. However, it is fraught with difficulty in proof where other watershed causes intervene, usually from upstream areas but not limited to such areas. For example, the effect of a dam and reservoir on ground water in the capillary fringe and on sediments as they tend to spread surface water and build up ground water in this fringe has not really been treated as yet.

The court established the new rule that property is taken in the Constitutional sense when inroads are made on an owner's use of his land to the extent that, as between private parties, a servitude would otherwise have been acquired by agreement or by the course of time. As a result, the 5th Amendment is now recognized in these cases as a basic rule of fairness (49).

The second significant part of the case was the court's answer to the government that a property owner is not obliged to sue when the flooding threatens his land, for to be required to do so would in effect force him to run many undue risks, such as several piecemeal law suits and also, premature evaluation of what property is really taken. By the same token, it could impose undue obligations on the government. This, of necessity, prolongs the final decision in semi-arid valleys.

As the overflow and erosion were merely parts of a continuous series of events in reality, there was no reason why the owner's suit should not be postponed until the river flow and channel situation had become reasonably well stabilized, such that a final, legal account of the damage could be struck off in full and thereby discharged. Even though the owner may have subsequently sold the property, or have done protection work against future damage, the damage originally inflicted on the property was, nevertheless, fully compensable. The big problem, however, is how to measure and ascertain, within the requirements of the law of evidence, the actual damage due to the structures or other obstruction.

In other words, the six-year statute of limitations does not begin to run when the initial interference with the use of property starts (when reservoir filling started or when the pool first filled and the land was partly submerged); it runs concurrently with the sequence of subsequent related physical events. Thus time, quantity of flow, and level of flow are recognized, as well as their

physical consequences. However, stabilization of a stream channel in high sediment producing areas often requires many years (50).

This is important in engineering survey and appraisal work because the statute of limitations can prevent legal recovery, once it has run its full course in time. The problem here is how to determine when and to what extent the statute begins to work against the damaged party and when it has terminated or run its full course. The injured party must have legal notice of a harmful invasion of his property before the statute begins to "run" against him. What is legal notice is a question of fact that varies with the circumstances.

The third established point was that the secondary injury from flooding, namely, erosion of the banks, was also fully compensable. Mere level of land needed for reservoir capacity does not determine what property has been taken. But what is or has been taken includes land washed away by the water, even though incidental to the main part of the land taken. When any such land is left in a condition to be of less value than before, the owner is entitled to full compensation for that loss. The government must pay for all it takes but no more (51).

The fourth established point was that the measure of the erosion damage was the cost of preventative measures, if this could be accomplished by prudent action of the property owner. Here, again, sound engineering practice comes into play. Even though the land were subsequently reclaimed by the owner, this would be no valid defense to his full recovery. It was the original taking that was compensable (52).

The fifth point was the rule that damage caused by intermittent flooding above the reservoir pool was also compensable (53). These points are reviewed here in some detail to help engineers and geologists understand the development of the law in the more complex cases so they can better anticipate what the courts may do in the future.

Taking of Property by Overflow, Erosion, and Sedimentation.—The next state in the development of the law took place in a high sediment producing area, the Colorado River, above Parker Dam and Havasu Reservoir, where the cause and effects are separated by many miles of channel. This was a forerunner to possible fuller development of the law on the Rio Grande River above Elephant Butte Reservoir. In these cases, the very important principle of the continuous series of related physical events was extended to sedimentation and its direct consequences.

But the Colorado suit dealt with damage to land with a fairly stable reservoir level below and a major water control dam above (Boulder), while the Rio Grande suit dealt with damage to railway facilities with a fluctuating reservoir level below and lack of major control dam above. Thus, in the Rio Grande situation the intervening watershed causes and the instability of the stream channel, due to fluctuating reservoir levels and runoff from above, complicated the factual situation; so did the presence of salt cedar, one of the most important and little analyzed influences in these cases.

The Cotton Land Company sued to recover compensation for damage to (taking of) its property midway between Hoover Dam (86 miles above) and Parker Dam (53.4 miles below). The land occupied an 8-mile frontage on the river, though in alternate sections for 16 miles, and extending back for 1 to $5\frac{1}{2}$ miles. It was ten miles above Topoch and partially opposite the town of Needles. The greater space, time, and quantity relationships are significant

here, compared to the earlier cases, dealing with more localized problems, mostly in the eastern states (See Fig. 1).

The river dropped sand and some coarse silt at the head of Havasu Lake, as a result of its scouring action below Hoover, beginning in 1939. This depo-

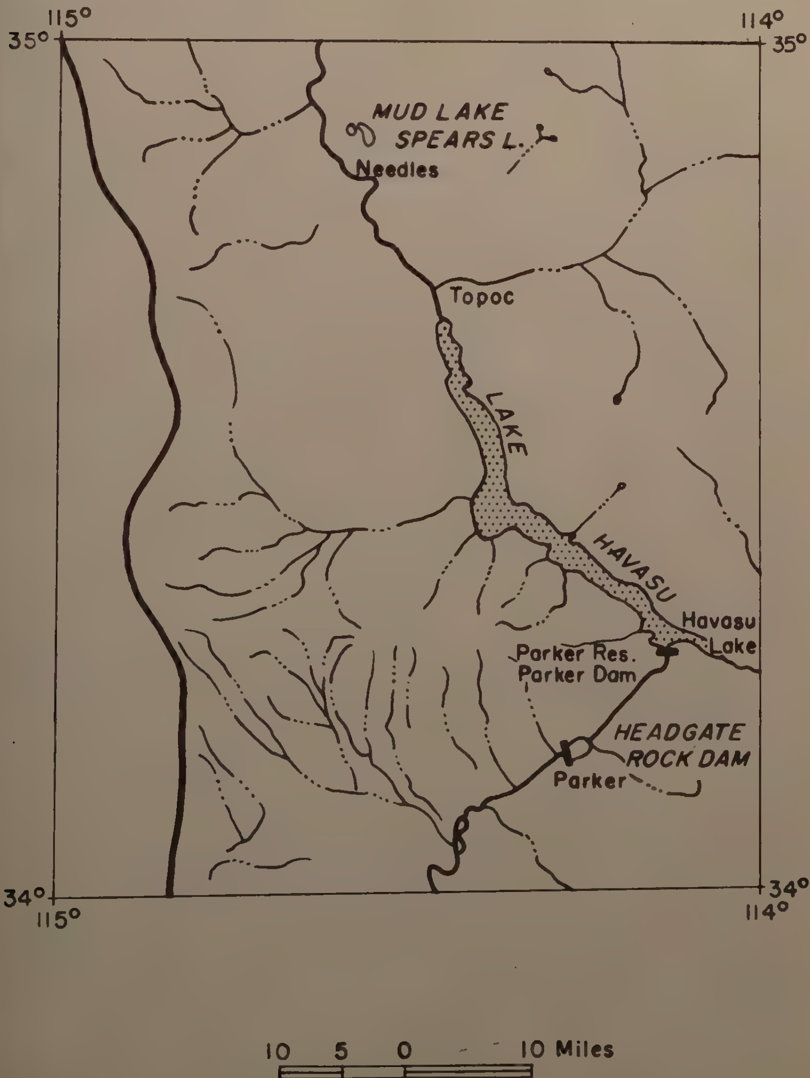

FIG. 1.—SEDIMENTATION ABOVE PARKER DAM AND HAVASU LAKE

sition "progressed" upstream, reaching Needles in 1939, and thereafter moved to a point 4 miles below the Company lands.

The sediment raised the river bed 9 ft, 3 miles above Topoch; 4 ft at Needles; and zero feet, 4 miles below the lands, as of the time of the suit.

Thus, the water level in the river channel was also raised and the flood waters overflowed the Company lands. This action progressed up the river concurrently with the sediment deposition in the channel. The process, as shown by the survey records in this case, seems to reflect growing stability of the river channel.

With heavy releases from Hoov Dam, the Company lands on the south were overflowed, and higher le of flows caused water to back into the sloughs on the north. There wer 00 acres flooded, and 2,000 acres isolated (stipulation by the parties). much was mentioned by the court about waterlogging of lands but this might be anticipated in other cases of this type, such as in the old suit on the Savanah River in Georgia (United States vs. Lynch, 188 U. S. 445, 1903), where established drainage ditches became no longer effective because of the dam. And, also, in the more recent case on Dardenne Creek, a non-navigable tributary of the Missouri River, in Missouri (United States vs. Kansas City Life Insurance Co., 339 United States 799, 807, 1950). The latter was a 5-4 decision in which one judge argued that the common enemy rule in Missouri would prevent recovery in a private suit, so why should the government pay?

The series of physical events here were: (1) construction of dam, (2) filling of lake, (3) deposition of sediment in and at head of lake, (4) movement of sediment deposition upstream therefrom, creating obstacles to flow and reduction in channel capacity, (5) rise in level of river flow in broad reaches, (6) overflow of banks, and (7) spreading of water over lands of the Company. The court does not seem to recognize, as a secondary cause, that the building of Hoover Dam and subsequent degradation of sediment below it by clear water affected deposition at Havasu. But these physical events all seem to stem from downstream causes. Upstream causes seem less significant here than in the Rio Grande situation. The events recognized by the court do not involve ground water and water loving plants, as such.

The court held there was no taking under the 5th Amendment on initial filling of the lake. But subsequent physical events caused by the obstruction led to the taking and set in motion the chain of related and connected events, such that the injury could not be considered legally remote. Furthermore, this question should not be confused with the issue of the extent to which the United States has consented to be sued. The court held that the remedy in obtaining compensation was as broad as the requirements of the Constitution (54).

The court also held that it was not necessary to find that the agents of the United States were aware that their acts would result in a taking, adding up to a promise to pay. And that compensation was due, even so, whether or not United States agents were aware of the effect of their acts. This is significant to engineers when the question of available scientific information and predictability methods are taken into consideration. The loss is compensable though it took considerable time to occur, resulting naturally from the improvements (55).

Taking of Property by Overflow, Erosion, Sedimentation, and Othe Causes.—The next step in development of the law has begun with the suit b the Santa Fe Railway against the government for damage alleged to have re sulted to the Company right-of-way and facilities by the construction of Ele phant Butte Dam and Reservoir. Although the physical situation is similar t that of the Colorado suit, there are important differences, such as to fluctu

ating reservoir levels and stream flow and the intervening upstream watershed causes. It appears that the consideration of the causes by the Commissioner was restricted mainly to engineering aspects of surface phenomena (56).

The findings of fact and recommendations for conclusion of law by the Commissioner were not fully adopted, as such, by the court, though most of the findings were officially accepted. The petition of the Company was dismissed without prejudice to a future suit (on the condition of finding an ascertainable harmful effect that is not legally too remote).

Perhaps this is a wait-and-see determination directed to possible future effects of government programs, future action of the river, and more adequate and certain records, as they relate to specific causes and issues. The relationship of salt cedar to sedimentation accumulation and flow damage and loss of water supply is an especially difficult and important factor in problems of this general type involving the consideration of soil moisture, capillary ground water, and sedimentation phenomena (57).

The determination of the court is that the suit of the Company was premature and that the plaintiff has failed to prove to the satisfaction of the court that the government dam and reservoir have increased or will increase the hazards to and expenses of maintaining the Company's properties. The court considered that the plaintiff had not proved with reasonable certainty, an ascertainable harmful effect in this regard.

It may be that the average annual sedimentation rates in this case tended to mask and obscure the cause and effect relationships shown by the cyclic fluctuations as carefully recorded at surveyed points. Variations of this type are sometimes more meaningful and certain than averages; both are abstractions, but averages are in a higher order of abstraction. Differences that make a difference are usually the more meaningful (58).

The court draws attention to the fact that the parties had agreed upon the rate of sedimentation, had the dam and reservoir not been built, of .27 ft per yr, at or near Bridge 1006-A. And, of course, the findings show the actual recorded rates, year by year and cycle by cycle. The general method of finding the facts set out by the Commissioner and that adopted by the court are available in the official report for study by engineers.

The significance of the Santa Fe Railway case, had it not been dismissed, could be the recognition that though damages by overflow, erosion, and sedimentation are inextricably intermingled with other damaging upstream influences. These include past overgrazing on the watershed, increased irrigation that depletes the sediment carrying capacity of the stream, salt cedar infestation that slows flood flows, and increased sedimentation on the flood plains. There is channel erosion after sediment loads are dropped, and artificial strictures in channels and valleys, produced by bridges, embankments and dikes). This would have been fully compensable.

However, it does not appear that any real consideration was given to the effect of the dam and reservoir on soil moisture and capillary ground water in the valley, as affecting the growth and spread of salt cedar, in combination with their combined effect on sediment accumulation in broad areas of the valley of low gradient and of critical sediment grade sizes with high moisture holding capacity in specific reaches.

Finding No. 31 recognizes that the delta is an ideal area for the growth of vegetation (in discussing salt cedar), and also the areas above Bridge 1006-A but does not relate the two broad localities ecologically in reference to the

dam and what took place sedimentation-wise between them. Had the suit been over lands and their water rights, this might have been a more vital factor.

These matters apparently led the Commissioner to the finding of an average "natural" rate of aggradation (parties agreed to this figure) and of an average artificial rate due to the dam. The interdependence of causes complicated the legal requirements of certainty and ascertainability.

The report of the case is presented, hereafter, in detail for discussion purposes because of its intensive treatment of the Rio Grande sedimentation problem.

GENERAL STATEMENTS

The Commissioner, in his report to the Court, reviewed the Cotton Land, Cress, Jacobs, and Dickinson cases (referred to herein) to show that they differ on the facts only as to degree, not as to kind; and that the resolution of the conflicting interests depends on the available facts and their careful quantitative evaluation.

The Cotton Land case is more directly in point, though Havasu Lake had more permanent levels than Elephant Butte; Boulder Dam served the preceding regulation purposes, and the cycles of sedimentation seem less erratic in the lower Colorado River. As to future cases of this type, engineers might familiarize themselves with the old Lynch case on the Savanah River, Georgia, the Sanguinette case on the Calaveras River, California, and the more recent Life Insurance case on the Missouri River tributary in Missouri. These treat waterlogging damage as a physical event resulting from dam construction in both navigable and non-navigable situations.

QUESTIONS RAISED

The Commissioner asked a series of questions preliminary to making his findings of fact. These are:

How large should flowage rights have been initially (in relation to periodic aggradation, fluctuating lake levels, and avulsion)?

When did the cause of action accrue (in relation to start of impoundment, intervening causes, and matters of legal notice to the complainant)?

What items should be compensated (in relation to maintenance and life of railway facilities, and other factors)?

Predictability of delta and reservoir influences?

Datum for surveys and evaluations?

(The Court omitted these questions.)

PERTINENT FINDINGS OF FACT

(No. 29) Causes and Effects of Aggradation.—The Court recognizes artificial restrictions, salt cedar, tributary stream erosion, increased upstream irrigation, overgrazing in past years, as well as the dam and reservoir as damage causes. It then focussed attention on Bridges 1002-A and 1006-A (with related facilities) in regard to injury suffered to the security and permanence of railway facilities. If the suit had been over damage to lands in the

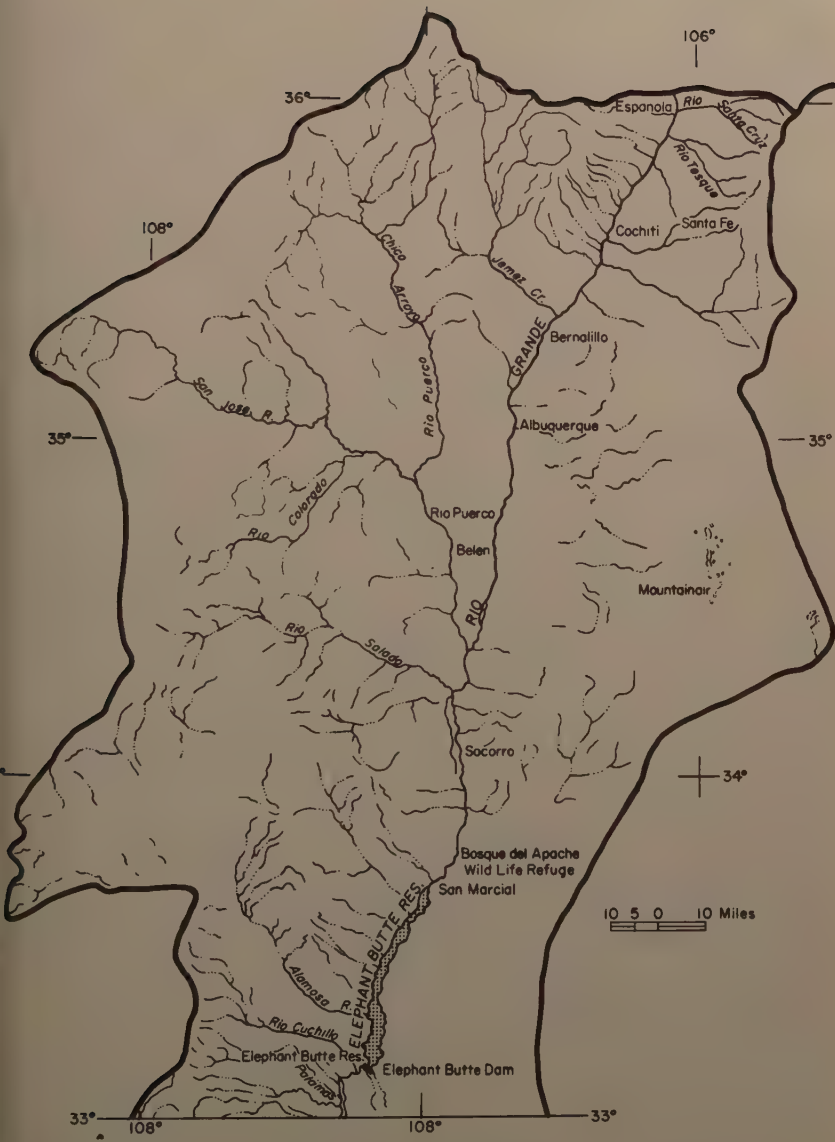

FIG. 2.—SEDIMENTATION ABOVE ELEPHANT BUTTE DAM AND RESERVOIR

San Marcial area, rather than over damage to railway facilities, the problem of cause and effect might have been more like the Cotton Land case.

(No. 30) Artificial Restrictions in Flow.—The Court recognized that the river channel has lost considerable capacity, the overflows are confined by dikes directed toward bridge openings, and these together reduce discharge capacities. The valley development has been altered but this is a natural process unless it can be shown to adversely affect the railway facilities. In spite of lack of statistics on total sedimentation, the sequence of restricted flow and increased velocity led to pressure on the dikes, aggradation above the embankments, and avulsion in flood seasons, thus adversely affecting railway property.

(No. 31) Salt Cedar Infestation.—The Court states that infestation has increased since 1930 in eight areas above Bridge 1006-A, and by 1947 became almost continuous in this stretch. It was also present below the bridge, but the Court does not define locations in respect to the delta.

This infestation causes sediment deposition on the flood plain more thar cottonwoods and willows, and the clear water does channel erosion work thereafter. The Court seems to absolve the plant as a factor contributing to legal damage.

The Court emphasizes this problem, especially, but barely touches this cause ecologically in relation to other related causes. The relationship o salt cedar to dam construction and sedimentation would seem to call for the combined judgment of engineers and ecologists because the plant has a critica moisture need, satisfied mainly by soil moisture conditions and by groun water conditions in the capillary fringe. Of critical importance to engineer concerned with the whole process is whether the dam and the sediments i caused to deposit upstream led to accumulation of silt size particles a suitable seed bed in certain localities and to rises in the water table an capillary fringe favorable to the germination, growth, and spread of sal cedar (59).

(No. 32) Tributary Erosion.—The Court recognized the clear Rio Grand River above Rio Chama, but heavy sediment carrier after Chama, Jemez Puerco, and Salado enter. This stems mainly from tributary channel erosio that started about 1895.

(No. 33) Irrigation and Overgrazing.—The tributary channel erosion sy chronized with a period of increased aridity, grazing, and use of water i irrigation between 1880 and 1915. The first two reduced water control on th tributary flood plains and the third removed a third to half the river flov especially in the summer months. It is presumed the government had leg notice of these events. What is legal notice in these circumstances to a priva individual or corporation is not clear.

(Nos. 34, 35, 36) Influence of Elephant Butte Reservoir.—The reader shou consult the series of tables included in the report for details on trends aggradation and degradation at various reaches in the stream channel over t years, especially the periods from 1931 to 1937 and 1941 to 1949. On thes facts, the Court recognized, as a matter of scientific knowledge, that load streams tend to deposit sediments at or near the point where they flow in lakes, reservoirs, and other still waters because of reduced velocity; a that greatest aggradation takes place at times of heavy flows, also causi scouring and degradation in certain reaches. Of course, the stream deposi sediments or overflows in areas of lower gradient that may be suitable f the growth of water loving plants.

The delta at the head of the reservoir takes time to form and is, itself, subject to erosion. But it compacts with age and then resists erosion more. With age it tends to cause a sediment wedge to build back upstream at greater rate than normal aggradation. As the reservoir level recedes by consumptive use (and low inflow), degradation sets in at the front of the delta and works back upstream seeking equilibrium. The influence of this factor differs significantly from the Colorado case. The Court recognized these scientific facts about the whole process and might have gone further in this regard if more reliable facts were available. This is the reason they are presented here.

(Nos. 37, 38, 39, 40, 41) Sediment Deposition.—In the first 4 yr to 5 yr after dam completion, more sediment was deposited than at any comparable period thereafter, especially in the 17-mile to 27-mile reach below Bridge 1006-A. This was the area occupied by the reservoir head, but such did not take place at the bridge itself.

From 1920 to 1925 heavy sediment deposition took place 10 miles to 18 miles and some 4 miles to 8 miles below the bridge. The average annual rate was about .84 ft then, rather than .27 ft as "normal." (Expert witnesses seemed not to be sure this was due to the reservoir.)

From 1925 to 1935 the reservoir and valley received added sediments in the reach from the bridge down to the dam, especially 14 miles to 20 miles below the bridge. This caused a wedge that later reached Bridge 1006-A. The court is more definite here by saying that the reservoir influence probably contributed to the increased aggradation from 1931 to 1937, together with other causes previously discussed. This was also the beginning of salt cedar infestation above the delta as shown by the record and other public reports to be referred to.

From 1937 to 1947 aggradation took place between 30 miles below and 10 miles above Bridge 1006-A. In 1941 and 1943 flood flows were high and the reservoir head extended closer to the bridge than at any time. Sediment filled the old erosion channels in the head delta. This caused aggradation to move upstream, involving also the area above the bridge. By 1937 salt cedar had spread into eight areas above Bridge 1006-A, and by 1947 into much of the valley below the bridge.

The Court takes up the quantitative aspects of the problem by comparing the average annual and the recorded rates, finding the latter higher for the period 1949 to 1954. Time and certain time periods seem to have been important in court recognition of the cause and the consequences, and of the application of the statute of limitations.

(No. 42) Effect of Reservoir.—The Court indicates that the reservoir may have accentuated the sediment process and interfered with the normal function of the river to dispose of its sediment load from the high contributing tributaries. This has been aggravated by natural and other artificial conditions.

The reservoir sediment rates predicted in 1916 were more than 50% in excess of what actually was deposited there. (This does not mean that there were not intervening influences that caused deposition above and prevented transport into the reservoir itself.) But the record leaves in conjecture the question of ultimate stabilization of sediments at or near the bridge.

(Nos. 43 to 50) Accrual of the Cause of Action; Control and Correction of Aggradation.—The Court then disposes of the contentions of the Railway Company and arrives at the conclusion that the cause of action could not have arisen until about 1949 when the aggradation had moved far enough upstream

to become reasonably well stabilized as a damaging factor at Bridge 1006-A, so as to bring the cause fully to the attention of the Company. Even then, the Company could not be sure it was permanent; for one thing, the silt survey was not available until then.

Legal notice, then, in the form of direct damage or the threat of damage, is necessary to give rise to a legal cause of action and start the statute of limitations to operate. But the completion of the legal process as cause, and its outlaw by reason of these limitations, is another matter. For here, again there is a sequence of physical events running concurrently with the statute. Had there been a major control dam above and stable reservoir level below together with information on the effect of the dam on ground water in the capillary fringe below the soil, the factual story might have been much different.

The control structures built on or planned for tributaries may help solve the sedimentation problem as it presently exists. But much depends on future floods, effect of improvements, and sedimentation effects. (These findings taken together, seem to have materially affected the present judgment of the Court.)

The writer's conclusion is that the law will eventually recognize all the harmful damage consequences from the building of dams, insofar as these can be proved as a sequence of related physical events and insofar as they involve the interests of the parties to the suit. Much depends on whether the damage is to lands and their water rights or to facilities and rights-of-way.

Time and reliable records may determine the issues. These should include vegetative as well as engineering considerations. Actually, the development of the law is only now unfolding, because the new rule of fairness was established and the new interpretation of what constitutes an obstruction to navigable waters. The same rules might well be considered by the respective states under their constitutions.

In this case, it is not clear by any means as to the effect of the dam on the growth of and spread of salt cedar through the media of sedimentation of fine materials in strategic locations and the rise in the capillary ground water fringe, if any. The tremendous expansion in salt cedar began about 1937 after establishment in the delta areas (Finding No. 31) (60), and this tends to coincide with the greatest increase in sedimentation (Finding No. 20). What was the concurrent position of the capillary fringe? Was this in any way due to the dam? What about deposition of fine sediments with high water-holding capacity? This presents a challenge to lawyers, engineers, and conservationists—a challenge of social engineering.

APPENDIX.—COURT CITATIONS AND REFERENCES

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act of lifting the river's mean level to the high watermark.") United States vs. Kansas City Life Insurance Co., 339 U. S. 799, 807, 812, 815, 1950.

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50. United States vs. Dickinson, supra, 748.

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FREE SURFACE FLOW IN HOMOGENEOUS POROUS MEDIUM

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SYNOPSIS

The damping of the unsteady flow through a dam or levee with horizontal underdrain, when the head behind it is raised considerably, is treated analytically and tested in a Hele-Shaw model. Slow and rapid rises from low level to final full level are considered. Numerical examples are included. The analytical and experimental results compare reasonably well.

INTRODUCTION

The essential idea used in the analysis is consideration of the unsteady flow as a time-dependent perturbation of the final steady flow. The unsteady potential $\phi(x, y, t)$ is expanded in a power series of $e^{-\lambda t}$, of the form

$$\phi(x, y, t) = \phi_0(x, y) + \phi_1(x, y)\, e^{-\lambda t} + O\left(e^{-2\lambda t}\right) \quad \ldots\ldots \quad (1)$$

in which $\phi_0(x, y)$ is the known steady-state potential, $\phi_1(x, y)$ denotes a perturbation potential, and

$$O(e^{-2\lambda t}) = \phi_2(x, y)\, e^{-2\lambda t} + \phi_3(x, y)\, e^{-3\lambda t} + \ldots \quad \ldots\ldots \quad (2)$$

Note.—Discussion open until December 1, 1961. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Hydraulics Division, Proceedings of the American Society of Civil Engineers, Vol. 87, No. HY 4, July, 1961.

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Each of the terms $\phi_n (x, y)\, e^{-n\lambda t}$ can be thought of as a perturbation term of its precursor in the series, and the present approach is limited to the computation of the first perturbation term $\phi_1 (x, y)\, e^{-\lambda t}$.

It is shown that ϕ_1 satisfies Laplace's equation $\nabla^2 \phi_1 = 0$ in a dimensionless hodograph plane. The free-boundary condition is linear but complicated, especially in the case of rapid level rises. This boundary condition contains the eigenvalue λ, which is found by the solution of a determinantal equation. In the analysis, the amplitude of the displacement of the free surface is left undetermined; only the mode of the motion and the eigenvalue are computed. N. Curle has obtained[2] a general technique for considering the unsteady development of steady two-dimensional jet and cavity flows. The approach used was that of expanding the velocity potential Φ in ascending powers of $e^{-\lambda t}$, so that

$$\Phi (x, y, t) = \Phi_0 (x, y) + \Phi_1 (x, y)\, e^{-\lambda t} + O\left(e^{-2\lambda t}\right) \quad \ldots\ldots \quad (3)$$

The unsteady free boundary was determined by normally displacing the steady state boundary by an amount

$$\delta (s, t) = \delta_1 (s)\, e^{-\lambda t} + O\left(e^{-2\lambda t}\right) \quad \ldots\ldots\ldots\ldots \quad (4)$$

In the present paper, the preceding technique is applied to a problem of free surface flow in a porous medium. Although the two flows have some essentially different features (gravity is neglected in the jet flow, kinetic energy in the seepage flow), the technique works equally well in both cases. The boundary-value problem arising in the present flow is somewhat more complicated. The trigonometric functions appearing in the free-boundary condition of the jet flow are replaced by algebraic functions, so that no recursion formula is available for the coefficients of the Fourier series solution of $\nabla^2 \Phi_1 = 0$. Instead, these coefficients result from the solution of a system of linear, homogeneous equations.

In a previous paper, the writer has treated[3] the case of slow level rises only, and has expressed the feeling that further theoretical investigations or experiments may lead to the relation between the unknown amplitude and the speed at which the water level behind the dam rises. Since Curle likewise has left the origin of the unsteady state phenomenon, and hence the amplitude, undetermined in the time, writer extended his own analytical work only to the case of rapid level rises. A Hele-Shaw model was used to check this analytical work and to determine the time origin of the damping. Thus theory and experiment combined provide a complete solution to the problem.

Notation.—The letter symbols adopted for use in this paper are defined where they first appear, in the illustrations or in the text, and are arranged alphabetically, for convenience of reference, in Appendix II.

ANALYTICAL APPROACH

Basic Assumptions.—It is assumed that the porous medium is completely saturated, homogeneous, and isotropic, and that the flow is everywhere lam-

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2 "Unsteady Two-dimensional Flows with Free Boundaries," by N. Curle, Proceedings, Royal Soc. of London (Series A), Vol. 235, 1956, p. 375.

3 "Unsteady Flow Through an Underdrained Earth Dam," by R. De Wiest, Journal of Fluid Mechanics, Vol. 8, Part 1, 1960, pp. 1-9.

inar. If the fluid is restricted to water of constant density and constant viscosity, then all the necessary conditions[4] for the existence of a velocity potential Φ are satisfied. Therefore

$$\Phi = K\,\phi = K\left(y + \frac{p}{\gamma}\right) \qquad (5)$$

and

$$K = k\frac{\gamma}{\mu} \qquad (6)$$

in which K is the hydraulic conductivity, dimensions $[L/T]$; k denotes permeability, depending on the medium alone, dimensions $[L^2]$; μ refers to viscosity of water; γ denotes the unit weight of water; ϕ is the hydraulic head; y denotes the elevation head; and p/γ is the pressure head. The term Φ has the dimensions $[L^2/T]$ and can be combined with the streamfunction Ψ, $[L^2/T]$, to form the complex potential function

$$W = (\Phi + i\,\Psi)\,(K) \qquad (7)$$

is assumed to be constant, and consequently so are k and the porosity ϵ of the medium. Effects of capillarity are neglected and Darcy's law in its simplest form can then be written

$$\vec{q} = -K \operatorname{grad} \phi \qquad (8)$$

and

$$u = -\frac{\partial \Phi}{\partial x},\; v = -\frac{\partial \Phi}{\partial y} \qquad (9)$$

The insertion of $\vec{q}$ into the continuity equation div $\vec{q} = 0$, leads to

$$\nabla^2 \phi = 0 \qquad (10)$$

Steady-state values are denoted by the subscript o, such as q_o; u_o; v_o; p_o; $\alpha = \tan^{-1}\frac{v_o}{u_o}$; ϕ_o; Φ_o; Ψ_o; W_o.

Steady Flow Equations.—It is common to derive the equations for the streamlines and equipotential lines by the mapping function $z = W_o^2$, in which z represents the physical plane and W_o is the complex potential. Because of numerical computations, and since the unsteady state problem is treated as a perturbation of the steady hodograph representation, several properties are demonstrated hereafter by the hodograph method.

The hodograph of the steady free surface is expressed[5] by

$$u_o^2 + v_o^2 + K\,v_o = 0 \qquad (11)$$

or

4 "The Theory of Groundwater Motion," by K. Hubbert, Journal of Geology, Vol. 48, 1960, p. 785.

5 "The Flow of Homogeneous Fluids Through Porous Media," by M. Muskat, J. W. Edwards, Inc., Ann Arbor, Mich., 1937, p. 302.

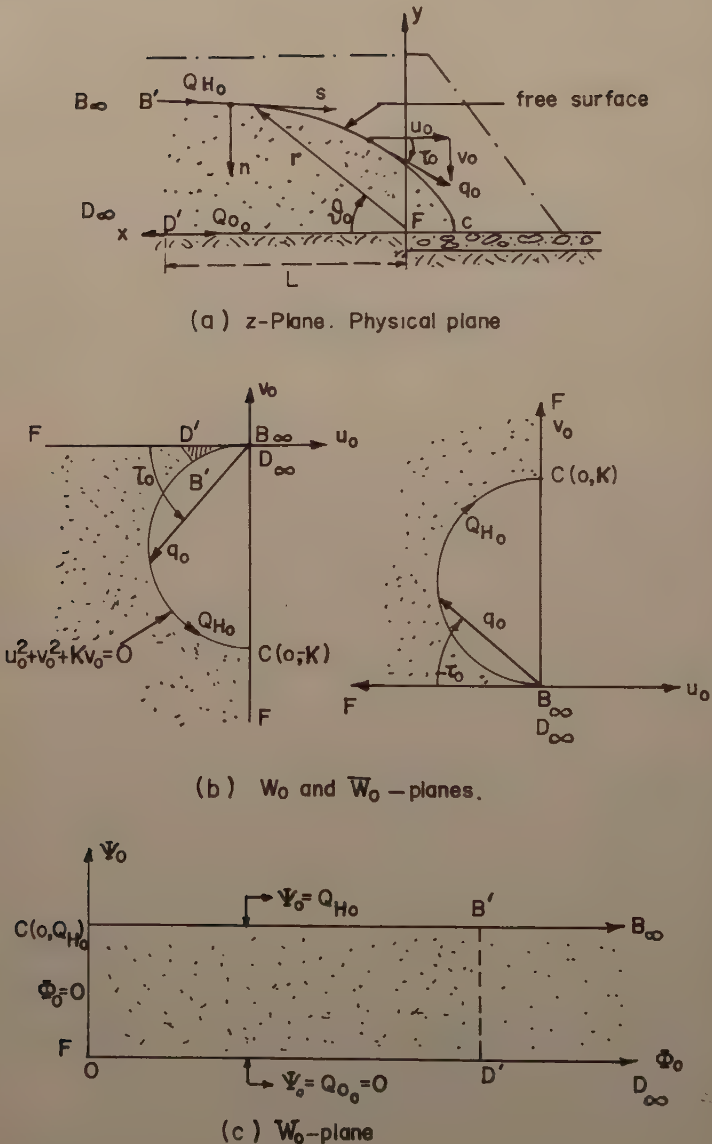

FIG. 1.—CONFORMAL MAPPINGS

$$\left(\frac{\partial \phi_o}{\partial x}\right)^2 + \left(\frac{\partial \phi_o}{\partial y}\right)^2 - \frac{\partial \phi_o}{\partial y} = 0 \quad \ldots \ldots \ldots \ldots \quad (12)$$

by the use of Eqs. 5 and 9. This equation is also derived explicitly as a limit case of the time dependent free surface equation. The mapping $z = W_o{}^2$ holds strictly only for the semi-infinite dam as does the following hodograph representation.

Streamlines and Equipotential Lines.—The z-plane, or physical plane, the velocity plane w_o and conjugate velocity-plane $\overline{w}_o$, and the complex potential plane W_o are represented in Fig. 1. The mapping from the $\overline{w}_o$-plane onto the W_o-plane is

$$W_o = -\frac{Q_{Ho}\,K}{\overline{w}_o} \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (13)$$

in which Q_{Ho}, $[L^2/T]$, is the steady seepage rate per unit width of dam for a head H behind the dam.

Along the free surface, u_o and v_o are negative as referred to the (x, y) coordinate system, whereas at the same point q_o is positive in the (s, n) system.

W_o and $\overline{w}_o$ are related so that

$$\frac{d\,W_o}{d\,z} = -\,(u_o - i\,v_o) = -\,\overline{w}_o \quad \ldots \ldots \ldots \ldots \ldots \quad (14)$$

Substitution of $\overline{w}_o$ from Eq. 13 into Eq. 14 and subsequent integration lead to

$$z = \int_o^{W_o} \frac{W_o\,dW_o}{K\,Q_{Ho}} = \frac{W_o{}^2}{2\,K\,Q_{Ho}} \quad \ldots \ldots \ldots \ldots \ldots \quad (15)$$

W_o may be found from Eq. 15 by the use of $z = r\,e^{i\theta_o}$ and by the omission of the negative sign, corresponding to the image flow of the physical flow through the dam, in the square root extraction. Therefore

$$W_o = \left(2\,K\,Q_{Ho}\,r\right)^{1/2} e^{i\theta_o/2} \quad \ldots \ldots \ldots \ldots \quad (16)$$

This leads to

$$\Phi_o = \left[K\,Q_{Ho}\,r\left(1 + \frac{x}{r}\right)\right]^{1/2} \quad \ldots \ldots \ldots \ldots \quad (17a)$$

and

$$\Psi_o = \left[K\,Q_{Ho}\,r\left(1 - \frac{x}{r}\right)\right]^{1/2} \quad \ldots \ldots \ldots \ldots \quad (17b)$$

The lines of constant potential are derived from Eqs. 17 by assigning to Φ_o an arbitrary constant value m, so that

$$y^2 = \frac{m^4}{K^2\,Q_{Ho}^2} - 2\,\frac{m^2}{K\,Q_{Ho}}\,x \quad \ldots \ldots \ldots \ldots \quad (18)$$

Similarly, the streamlines have the following equation

$$y^2 = \frac{n^4}{K^2\, Q_{Ho}^2} + 2\,\frac{n^2}{K\, Q_{Ho}}\, x \quad \ldots\ldots\ldots\ldots\ldots \quad (19)$$

in which n is an arbitrary constant value of Ψ_o in Eqs. 17. In particular, the equation of the free surface follows for $\Psi_o = n = Q_{Ho}$. Therefore

$$y^2 = \frac{Q_{Ho}^2}{K^2} + 2\,\frac{Q_{Ho}}{K}\, x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (20)$$

Eqs. 18 and 19 are represented by two nests of orthogonal, confocal parabolas. The coordinates of C in Fig. 1(a) are

$$\left(-\frac{Q_{Ho}}{2\,K}, o\right)$$

The steady seepage Q_{Ho}, per unit width of the dam, is found by the insertion of the coordinates of B' (L, H) into Eq. 20, therefore

$$Q_{Ho} = K\left[\left(H^2 + L^2\right)^{1/2} - L\right] \quad \ldots\ldots\ldots\ldots \quad (21)$$

The velocity along the impervious bottom may be computed by

$$v_r = -\frac{1}{r}\,\frac{\partial \Psi_o}{\partial \theta_o} = -\left(\frac{K\, Q_{Ho}}{2\, r}\right)^{1/2} \cos\frac{\theta_o}{2} \quad \ldots\ldots\ldots\ldots \quad (22)$$

Along this boundary, $\theta_o = 0$ and $r = x$, so that

$$\left(v_r\right)_{FD\infty} = -\left(\frac{K\, Q_{Ho}}{2\, x}\right)^{1/2} \quad \ldots\ldots\ldots\ldots\ldots \quad (23)$$

This can be written as

$$\left(v_r\right)_{FD\infty} = -\frac{K}{\sqrt{2}}\left\{\left[1 + \left(\frac{H}{x}\right)^2\right]^{1/2} - 1\right\}^{1/2} \quad \ldots\ldots\ldots\ldots \quad (24)$$

From Eq. 24 it follows that $v_{r,F} \to -\infty$ and $v_{r,D\infty} \to 0$.

Equivalent Expressions in Function of (q_o, τ_o) instead of (r, θ_o).—A combination of Eqs. 13 and 15 gives

$$z = \frac{1}{2}\,\frac{K\, Q_{Ho}}{\overline{w}_o^{\,2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (25)$$

Because $\overline{w}_o = q_o\, e^{-i\,\tau_o}$, it is derived from Eq. 25 that along any streamline

$$x = \frac{1}{2}\, K\, Q_{Ho}\, \frac{\cos 2\,\tau_o}{q_o^{\,2}} \quad \ldots\ldots\ldots\ldots\ldots \quad (26a)$$

and

$$y = \frac{1}{2} K Q_{Ho} \frac{\sin 2 \tau_o}{q_o^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (26b)$$

In particular, along the free surface

$$q_o^2 = - K v_o \ (11)$$

and because $v_o = q_o \sin \tau_o$, it follows that

$$q_o = - K \sin \tau_o \quad \ldots\ldots\ldots\ldots\ldots\ldots (27)$$

and the parametric equations become

$$x = \frac{1}{2} \frac{Q_{Ho} \cos 2 \tau_o}{K \sin^2 \tau_o} \quad \ldots\ldots\ldots\ldots\ldots\ldots (28a)$$

and

$$y = \frac{1}{2} \frac{Q_{Ho} \sin 2 \tau_o}{K \sin^2 \tau_o} \quad \ldots\ldots\ldots\ldots\ldots\ldots (28b)$$

Expressions for Φ_o and Ψ_o are further derived from Eq. 13 modified to

$$W_o = - \frac{K Q_{Ho} w_o}{q_o^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (29)$$

so that

$$\Phi_o = - \frac{K Q_{Ho} u_o}{q_o^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (30a)$$

and

$$\Psi_o = - \frac{K Q_{Ho} v_o}{q_o^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (30b)$$

If Eqs. 30 are squared, then

$$\Phi_o^2 + \Psi_o^2 = \frac{K^2 Q_{Ho}^2}{q_o^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (31)$$

The difference $\Phi_o^2 - \Psi_o^2$ is found by a direct combination of Eqs. 15 and 26, which gives

$$\Phi_o^2 - \Psi_o^2 = \left(\frac{K Q_{Ho}}{q_o}\right)^2 \cos 2 \tau_o \quad \ldots\ldots\ldots (32a)$$

and

$$\Phi_o \ \Psi_o = \frac{1}{2}\left(\frac{K\ Q_{Ho}}{q_o}\right)^2 \sin 2\ \tau_o \quad \ldots\ldots\ldots\ldots (32b$$

From Eqs. 31 and 32 one finds along any streamline

$$\Phi_o = -\ \frac{K\ Q_{Ho} \cos \tau_o}{q_o} \quad \ldots\ldots\ldots\ldots (33a$$

and

$$\Psi_o = -\ \frac{K\ Q_{Ho} \sin \tau_o}{q_o} \quad \ldots\ldots\ldots\ldots (33b$$

In particular, along the free surface from Eq. 27

$$\Phi_o = Q_{Ho} \text{ cotg } \tau_o$$

and

$$\Psi_o = Q_{Ho} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (34$$

Dam of Finite Length - Estimate of Error Involved.—Considerations of semi infinite dams have no value unless they can be applied, with some error, t dams of finite length, long enough however to be treated as infinite extent fo present purposes. Similar simplifications have been made in the original in vestigations of beams on elastic foundations. Mathematically it would mea the mapping of the F_∞ D' B' C F_∞ domain of the w_o plane of Fig. 1 on the fi nite strip F C B' D' of the W_o plane instead of the F_∞ B C F_∞ domain on th semi-infinite strip of the W_o plane, using the same mapping function in bot cases. It seems extremely difficult to compute the error involved in doing s It is clear though that the approximation will most affect the flow in the vicini of the inflow face B' D'. One expects little influence near the free surface an in the section of the drain, regions in which the main interest of this pape lies. Furthermore, while a sloping straight line could have been substitute for the parabolic shape of the inflow face, the line B' D' has been adopted view of the ease of constructing a Hele-Shaw model with vertical inflow fac An attempt to estimate the error involved follows. This attempt is not rigo ous, but one should accept the fact that an error in the estimate of an error not significant. Along B' D (Fig. 2), in the case of the parabolic inflow fac $\Phi_D = \left(2\ K\ Q_{Ho}\ L^*\right)$ from Eq. 17. If the reservoir behind the dam is assume to be in hydrostatic conditions everywhere, then also Φ_D = K H, so that

$$L^* = \frac{K\ H^2}{2\ Q_{Ho}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3$$

This is not an exact formula, because hydrostatic conditions will not strict prevail everywhere along B' D. If, on the other hand, the potential in D is a proximated by assigning to D the abscissa L instead of L* in Eq. 17, then pseudo-potential $\Phi'_D = \left(2\ K\ Q_{Ho}\ L\right)^{1/2}$ would appear. But since B' is on t free surface, 2 K Q_{Ho} L may be derived from Eq. 20 and inserted to give

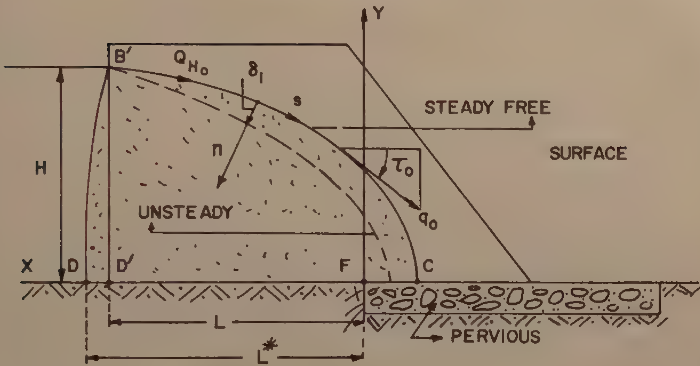

FIG 2.—UNDERDRAINED EARTH DAM

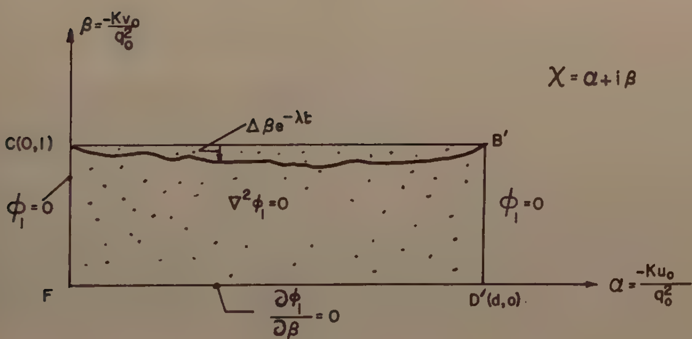

FIG. 3.—HODOGRAPH PLANE $X = -\frac{K}{\overline{W}_0}$

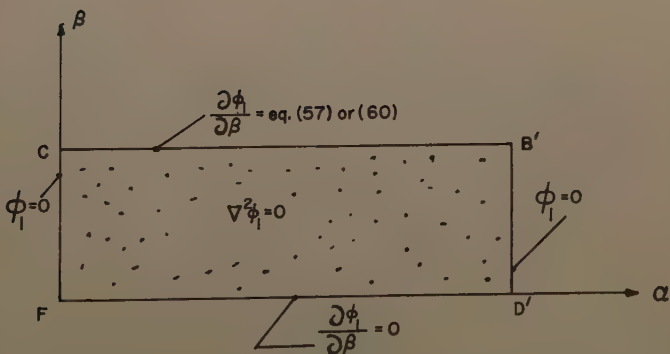

FIG. 4 —BOUNDARY CONDITIONS

$$\Phi'_D = K\,H \left(1 - \frac{Q_{Ho}^2}{K^2\,H^2}\right)^{1/2} \quad \ldots\ldots\ldots\ldots (36)$$

The term Φ'_D may be substituted for Φ_D if $\frac{Q_{Ho}^2}{K^2\,H^2}$ is sufficiently small. A value of 0.95 Φ_D is attained by Φ'_D, provided $Q_{Ho} \leq 0.312$ K H or $\frac{H}{L} \leq 0.69$, whereas a value of 0.945 Φ_D is rendered by Φ'_D, provided $Q_{Ho} \leq 0.33$ K H, or $\frac{H}{L} \leq 0.74$, using Eqs. 36 and 21. These numerical values show that the error in Φ_D is not sensitive to the $\frac{H}{L}$ ratio of the dam. The slope of the free surface in B' would of course be zero for the dam with vertical inflow face. The same slope, in the case of the dam with parabolic inflow face, would be derived from

$$(\mathrm{tg}\ \tau_o)_{B'} = \frac{\Psi_{B'}}{\Phi_{B'}} = \frac{Q_{Ho}}{K\,H} \quad \ldots\ldots\ldots\ldots\ldots (37)$$

For a dam with $\frac{H}{L} = 0.74$,tthis leads to $\tau_o = 18°16'$ and to $q_{B'} = 0.313$ K from Eq. 27 and $u_{D'} = -0.346$ K from Eq. 24. Points B' and D' are represented on the hodograph of Fig. 1. The curvilinear triangle B D' B' will shrink as the $\frac{H}{L}$ ratio of the dam decreases.

Unsteady Flow.—

Statement of Problem and Outline of Method.—The problem considered is that of a homogeneous dam or earth embankment with a horizontal underdrain, behind which the level of a reservoir is raised to a depth H (Fig. 2). It is to be expected that some time will elapse before steady flow will be reached after the head H has been kept at a constant value. The purpose herein is to analyze the flow when it approaches its final steady state.

Because the compressibility of the medium in unconfined flow can be neglected, Eqs. 5, 8, 9, and 10 are valid as well for $\phi_o(x, y)$ as for $\phi(x, y, t) = y(x, t) + 1/\gamma\, p(x, y, t)$. The expansion of $\phi(x, y, t)$ in a power series of $e^{-\lambda t}$, of the form

$$\phi(x, y, t) = \phi_o(x, y) + \phi_1(x, y)\, e^{-\lambda t} + O\ e^{-2\lambda t} \quad \ldots\ldots (38)$$

implies that $\phi_1(x, y)$ is a harmonic function, if $\phi_o(x, y)$ represents the known steady potential. In Eq. 38, λ represents an eigenvalue to be determined. The perturbation potential $\phi_1(x, y)$ remains harmonic under a conformal transformation. Because the steady hodograph can be reduced to a simple configuration, it is quite natural to look for a solution of $\nabla^2\,\phi_1 = 0$ in such a transformed hodograph plane if the basic equation of motion of the unsteady free surface is transferred to the steady boundary. As in the jet flow, the unsteady boundary may be found by normally displacing the steady-state boundary by an amount

$$\delta(x, y, t) = \delta_1(x, y)\, e^{-\lambda t} + O\left(e^{-2\lambda t}\right) \quad \ldots\ldots\ldots (39)$$

Boundary Conditions at the Free Surface.—Along the free surface, atmospheric pressure is assumed to be constant in time and space. Let y = h (x, t)

be the equation of the free surface. Because ϕ is a potential, the constant p/γ at the free surface can be omitted. The first free boundary condition can then be stated as

$$y = h\,(x, t) \qquad (40a)$$

or

$$\phi\,(x, y, t) = h\,(x, t) \qquad (40b)$$

From Eqs. 40, it cannot be deduced that $\frac{\partial \phi}{\partial x}$ would be equal to $\frac{\partial h}{\partial x}$. Indeed, at the free boundary

$$\phi\,(x, y, t) = h\,(x, t) + \frac{1}{\gamma}\,p\,(x, y) \qquad (41)$$

and differentiation at constant y and constant t leads to

$$\frac{\partial \phi}{\partial x} = \frac{\partial h}{\partial x} + \frac{1}{\gamma}\,\frac{\partial p}{\partial x} \qquad (42)$$

$\frac{\partial p}{\partial x} = 0$ only for a horizontal free surface and negligible only in case of a slightly sloping free surface, but $\frac{\partial p}{\partial x}$ is significant when the curvature of the free surface is significant. The second free boundary condition is that of a true boundary or a bounding surface.[6] Let y(t) and x(t) be the parametric coordinates of a fluid particle P. The concept of a true boundary is based on the fact that a particle at the surface cannot leave this surface. Hence the coordinates of P, brought to the surface, may be inserted in Eqs. 40 to make the first boundary condition an identity in t. Therefore

$$y(t) \equiv h\,[x(t), t] \qquad (43)$$

Total differentiation of Eq. 43 leads to

$$\frac{dy}{dt} = \frac{\partial h}{\partial x}\,\frac{dx}{dt} + \frac{\partial h}{\partial t} \qquad (44)$$

Because $\frac{dx}{dt} = \frac{u}{\epsilon}$, $\frac{dy}{dt} = \frac{v}{\epsilon}$, in which ϵ is the porosity of the medium, substitution of Eq. 9 in Eq. 44 leads to

$$\frac{\epsilon}{K}\,\frac{\partial h}{\partial t} = \frac{\partial \phi}{\partial x}\,\frac{\partial h}{\partial x} - \frac{\partial \phi}{\partial y} \qquad (45)$$

Eq. 45 is the basic equation of motion of the unsteady free surface. It can be transformed conveniently by computing $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial t}$. This is accomplished by

6 "Hydrodynamics," by H. Lamb, Cambridge University Press, Cambridge, England, 1932, p. 7.

differentiating Eq. 40 once at constant x and once at constant t. Along the surface, at constant time, Eq. 40 becomes y = h(x) and therefore

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial x} = \frac{\partial h}{\partial x} \qquad \text{(46a)}$$

or

$$\frac{\partial \phi}{\partial x} = \frac{\partial h}{\partial x}\left(1 - \frac{\partial \phi}{\partial y}\right) \qquad \text{(46b)}$$

At a given x, the boundary moves with time so that

$$\frac{\partial \phi}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \phi}{\partial t} = \frac{\partial h}{\partial t} \qquad \text{(47a)}$$

or

$$\frac{\partial \phi}{\partial t} = \frac{\partial h}{\partial t}\left(1 - \frac{\partial \phi}{\partial y}\right) \qquad \text{(47b)}$$

Eqs. 46 and 47 can be solved for $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial t}$. Substitution of the latter in Eq. 45 leads finally to

$$\frac{\epsilon}{K}\frac{\partial \phi}{\partial t} = \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 - \frac{\partial \phi}{\partial y} \qquad \text{(48)}$$

The steady state hodograph Eq. 12 is a special case of Eq. 48. Eq. 48 has been derived previously in a different way.[7,8]

Formulation of the Boundary Value Problem.—Consider the dimensionless hodograph plane of Fig. 3

$$\chi = \alpha + i\,\beta = -\frac{K}{\overline{w}_o} \qquad \text{(49)}$$

in which $\overline{w}_o = u_o - i\,v_o$; χ is simply related to W_o by $W_o = Q_{Ho}\,\chi$

The abscissa $d = \kappa\ K\,H/Q_{Ho}$, in which κ is a constant depending on the ratio $\frac{H}{L}$ of the dam, can de derived from these considerations. From Eq. 23

$$v_{r,D} = u_D = \left(K\,Q_{Ho}/2\,L^*\right)^{1/2} = -\frac{Q_{Ho}}{K} \qquad \text{(50)}$$

when the value of L*, Eq. 35, is inserted. Therefore $\alpha_D = \frac{K\,H}{Q_{Ho}}$. In a similar way, it is found that for the dam with the ratio $\frac{H}{L} = 0.74$, $\alpha_{D'} = 0.945\,\frac{K\,H}{Q_{Ho}}$. Again the value of α_D may be assigned to $\alpha_{D'}$ with a 5.5% error. This is also

7 "Teoria Dvizjenia Gruntovich Vod," by P. Polubarinova-Kotchina, Moscow, 1952, p. 619.

8 "Nouvelles Méthodes de Calcul Pratique des Ecoulements de Filtration non Permanents à Surface Libre," by M. Schneebeli, La Houille Blanche, 1953, No. B Special, p. 763.

evident since $\phi_o = \frac{Q_{Ho}}{K}\alpha$. The boundary conditions on FD', D' B', and FC are as follows

$\frac{\partial \phi_1}{\partial \beta} = 0$ along FD', because this is a streamline;

$\phi_1 = 0$ along FC, because the underdrain liquid is assumed to follow the hydrostatic law; and

$\phi_1 = 0$ along B'D', because of the formulation of the problem.

The rest of this subdivision will be spent on the derivation of an expression for $\frac{\partial \phi_1}{\partial \beta}$ on the boundary CB', and with this derivation, the problem will be set up mathematically. Let $\phi(x, y, t)$ satisfy Eq. 48. Then it follows that, at the unsteady free surface

$$-\frac{\epsilon}{K}\lambda\,\phi_1\,e^{-\lambda t} = \left(\frac{\partial \phi_o}{\partial x}\right)^2 + \left(\frac{\partial \phi_o}{\partial y}\right)^2 - \frac{\partial \phi_o}{\partial y} + \left(2\,\frac{\partial \phi_o}{\partial x}\,\frac{\partial \phi_1}{\partial x} + 2\,\frac{\partial \phi_o}{\partial y}\,\frac{\partial \phi_1}{\partial y} - \frac{\partial \phi_1}{\partial y}\right)e^{-\lambda t} + O\left(e^{-2\lambda t}\right) \qquad (51)$$

This condition can be transferred to one at the steady free boundary in the hodograph plane. Assume that, corresponding to the displacement $\delta_1(x, y)\,e^{-\lambda t}$ of the boundary in the physical plane, there is a displacement $\Delta\beta\,e^{-\lambda t}$ in the hodograph plane. Under this assumption, Eq. 51 can be transferred to the steady boundary by addition of a term $-\Delta\beta\,e^{-\lambda t}$. Once at the steady boundary, and only then, $\left(\frac{\partial \phi_o}{\partial x}\right)^2 + \left(\frac{\partial \phi_o}{\partial y}\right)^2 - \frac{\partial \phi_o}{\partial y} = 0$, and, ignoring $O\left(e^{-2\lambda t}\right)$, Eq. 51 reduces to

$$-\frac{\epsilon}{K}\lambda\,\phi_1 = 2\,\frac{\partial \phi_o}{\partial x}\,\frac{\partial \phi_1}{\partial x} + 2\,\frac{\partial \phi_o}{\partial y}\,\frac{\partial \phi_1}{\partial y} - \frac{\partial \phi_1}{\partial y} - \Delta\beta \qquad (52)$$

Now

$$\dot{q}_s\,\frac{\partial \phi_1}{\partial s} = -K\left(\frac{\partial \phi_1}{\partial x}\,\frac{\partial \phi_o}{\partial x} + \frac{\partial \phi_1}{\partial y}\,\frac{\partial \phi_o}{\partial y}\right) \qquad (53)$$

Recall that $W_o = \Phi_o + i\,\Psi_o = Q_{Ho}\,\alpha + i\,Q_{Ho}\,\beta$, so that along the steady free surface

$$\frac{\partial}{\partial s} = -q_o\,\frac{\partial}{\partial \Phi_o} = -\frac{q_o}{Q_{Ho}}\,\frac{\partial}{\partial \alpha} \qquad (54a)$$

$$\frac{\partial}{\partial n} = -q_o\,\frac{\partial}{\partial \Psi_o} = -\frac{q_o}{Q_{Ho}}\,\frac{\partial}{\partial \beta} \qquad (54b)$$

and

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \Phi_o}\,\frac{\partial \Phi_o}{\partial y} = -\frac{v_o}{Q_{Ho}}\,\frac{\partial}{\partial \alpha} \qquad (54c)$$

Then Eq. 52 reduces by means of Eqs. 53 and 54 to

$$-\frac{\epsilon}{K}\lambda\,\phi_1 = \frac{2}{K}\,\frac{q_o^{\,2}}{Q_{Ho}}\,\frac{\partial\phi_1}{\partial\alpha} + \frac{v_o}{Q_{Ho}}\,\frac{\partial\phi_1}{\partial\alpha} - \Delta\beta \quad \ldots\ldots\ldots \quad (55)$$

with

$$\Delta\beta = \frac{\partial\beta}{\partial n}\,\delta_1 = -\frac{q_o}{Q_{Ho}}\,\delta_1 \quad \ldots\ldots\ldots\ldots\ldots \quad (56)$$

and

$$q_o^{\,2} = -K\,v_o \quad (11)$$

Eq. 55 reduces to

$$\frac{\epsilon}{K}\lambda\,\phi_1 - \frac{v_o}{Q_{Ho}}\,\frac{\partial\phi_1}{\partial\alpha} + \frac{q_o}{Q_{Ho}}\,\delta_1 = 0 \quad \ldots\ldots\ldots\ldots \quad (57)$$

A second relation between δ_1 and ϕ_1 can be found to hold at the free surface by mathematically expressing that this surface is a material line. The unsteady free surface is not a streamline, but has a velocity component normal to the boundary. This component is due to the following factors:

1. To changes in $\delta_1\,(x, y)\,e^{-\lambda t}$ because of the exponential decrement with time. These changes, at any (x, y) are expressed as

$$\delta_1\,(x, y)\,e^{-\lambda(t+\Delta t)} - \delta_1\,(x, y)\,e^{-\lambda t} \approx -\lambda\,\Delta t\,\delta_1\,e^{-\lambda t}$$

The corresponding displacement of the boundary, of order $e^{-\lambda t}$, is equal to

$$-\frac{K}{\epsilon}\,\frac{\partial\phi_1}{\partial n}\,e^{-\lambda t}\,\Delta t$$

It follows that

$$\frac{K}{\epsilon}\,\frac{\partial\phi_1}{\partial n} = \lambda\,\delta_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (58)$$

2. To changes in $\delta_1\,e^{-\lambda t}$ because s changes as the fluid particles are convected along the free surface. They can be expressed as

$$\left[\delta_1\left(s + \frac{q_o}{\epsilon}\,\Delta t\right) - \delta_1\,(s)\right] e^{-\lambda t} \approx \frac{q_o}{\epsilon}\,\frac{d\,\delta_1}{d\,s}\,\Delta t\;e^{-\lambda t}$$

so that

$$K\,\frac{\partial\phi_1}{\partial n} = -q_o\,\frac{d\,\delta_1}{d\,s} \quad \ldots\ldots\ldots\ldots\ldots \quad (59)$$

The writer[9] has computed the case in which the component given by Eq. 5 would be small compared to that given by Eq. 58, and therefore could be neg

[9] "Unsteady-State Phenomena in the Flow Through Porous Media," by R. De Wiest Tech. Report No. 3, Civ. Engrg. Dept., Stanford Univ., 1959, p. 54.

lected. Under this assumption, the free boundary condition is derived as follows. Application of Eq. 54 to Eq. 58 leads to

$$\frac{K}{\epsilon}\,\frac{q_o}{Q_{Ho}}\,\frac{\partial \phi_1}{\partial \beta} = -\lambda\,\delta_1 \quad \ldots\ldots\ldots\ldots\ldots \quad (60)$$

Elimination of δ_1, from Eqs. 57 and 60 gives

$$\frac{\partial \phi_1}{\partial \beta} = \frac{\epsilon^2}{K^2}\,\frac{Q_{Ho}^2}{q_o^2}\,\lambda^2\,\phi_1 - \frac{\epsilon}{K}\,\frac{Q_{Ho}}{q_o^2}\,\lambda\,v_o\,\frac{\partial \phi_1}{\partial \alpha} \quad \ldots\ldots \quad (61)$$

Note that

$$\alpha = -\frac{K\,u_o}{u_o^2 + v_o^2},\quad \beta = -\frac{K\,v_o}{u_o^2 + v_o^2},\quad q_o^2 = \frac{K^2}{\alpha^2 + \beta^2} \quad \ldots\ldots\ldots \quad (62)$$

and introduce $a = \epsilon\,Q_{Ho}/K^2$, dimension $[T]$, to find at the free surface ($\beta = 1$), therefore:

$$\left(\frac{\partial \phi_1}{\partial \beta}\right)_{\beta = 1} = (a\,\lambda)^2\,(1+\alpha^2)\,\phi_1 + (a\,\lambda)\,\frac{\partial \phi_1}{\partial \alpha} \quad \ldots\ldots \quad (63)$$

It is convenient to consider $(a\,\lambda)$ as a dimensionless parameter. In case the component given by Eq. 59 cannot be neglected

$$\frac{K}{\epsilon}\,\frac{\partial \phi_1}{\partial n} = -\frac{q_o}{\epsilon}\,\frac{d\,\delta_1}{ds} + \lambda\,\delta_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (64)$$

or, by the use of Eq. 54

$$\frac{K}{\epsilon}\left(-\frac{q_o}{Q_{Ho}}\,\frac{\partial \phi_1}{\partial \beta}\right) = \frac{q_o^2}{\epsilon\,Q_{Ho}}\,\frac{\partial \delta_1}{\partial \alpha} + \lambda\,\delta_1 \quad \ldots\ldots\ldots\ldots \quad (65)$$

Insert the value of δ_1, from Eq. 57 into Eq. 65 and use Eq. 62 to find

$$\left(\frac{\partial \phi_1}{\partial \beta}\right)_{\beta = 1} = \left[(a\lambda)\,\alpha\,\left(1+\alpha^2\right)^{-1} + (a\,\lambda)^2\left(1+\alpha^2\right)\right]\phi_1$$

$$+\left[2\,(a\,\lambda) - \alpha\,(1+\alpha^2)^{-2}\right]\frac{\partial \phi_1}{\partial \alpha} + (1+\alpha^2)^{-1}\,\frac{\partial^2 \phi_1}{\partial \alpha^2} \quad \ldots\ldots \quad (66)$$

To conclude, one has to solve $\nabla^2\,\phi_1 = 0$ in the α, β plane with the boundary conditions as shown in Fig. 4.

Nature of the Boundary Value Problem.—Kantorowitsch and Krylow[10] give a brief analysis of what they call the general boundary value problem of potential theory. They state this problem as follows: Find a function ϕ which is harmonic inside a region R and which satisfies the condition

10 "Näherungsmethoden der höheren Analysis," by Kantorowitsch and Krylow, VEB Deutscher Verlag der Wissenschaften, Berlin, 1956, p. 530.

$$f \frac{\partial \phi}{\partial n} + l \frac{\partial \phi}{\partial s} + m \phi = \xi \quad \ldots\ldots\ldots\ldots\ldots \quad (67)$$

at the boundary L of this region. In Eq. 67, f, l, m, ξ, are given real functions of the independent variables along the boundary; n is the inner normal. The Dirichlet problem and Neumann problem arise from this more general problem when f, l, m, ξ are replaced in an appropriate way by constants or zeros. Hilbert's problem arises in the same way when m = 0 in Eq. 67. When the region is a circle or any simply connected domain, it can be reduced to the solution of an integral equation. Kantorowitsch and Krylow state that the general problem in which $m \neq 0$ can also be brought back to the solution of an integral equation, merely by reducing it to Hilbert's problem, although this reduction is tedious and difficult. They limit their considerations to the solution of Laplace's equation in the circle with the boundary condition

$$f \frac{\partial \phi}{\partial n} + m \phi = \xi \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (68)$$

Even this case is complicated and its understanding requires an advanced knowledge of the theory of integral equations. The present problem is of the nature of the general boundary value problem. It is somewhat complicated by the fact that the unknown eigenvalue appears in the boundary condition.

Existence of a Solution.—If there is a solution ϕ_1, it must satisfy the condition

$$\int_0^d \left(- \frac{\partial \phi_1}{\partial \beta}\right)_{\beta = 1} d\alpha + \int_0^1 \left(- \frac{\partial \phi_1}{\partial \alpha}\right)_{\alpha = 0} d\beta$$

$$+ \int_0^1 \left(\frac{\partial \phi_1}{\partial \alpha}\right)_{\alpha = d} d\beta = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (69)$$

expressing that the perturbation flow must satisfy the continuity requirement that the mass flux of the perturbation flow entering the fluid-filled region through the free surface equals that leaving through the entrance surface and the drain. This perturbation flow vanishes like $e^{-\lambda t}$ and is superimposed on the steady flow. Separation of variables for $\nabla^2 \phi_1 = 0$ in the strip of Fig. 4 leads to a solution of the form

$$\phi_1 = \sum_{n=1}^{\infty} A_n \cosh \frac{n \pi}{d} \beta \sin \frac{n \pi}{d} \alpha \quad \ldots\ldots\ldots\ldots \quad (70)$$

This solution satisfies the boundary conditions along FD', D' B', and FC; A_n is left to be determined by the boundary condition along CB'.

Free boundary condition Eq. 63: It follows from Eq. 69 and some integrations that the dimensionless parameter $a\lambda$ now must satisfy the condition

$$(a\lambda)^2 = \frac{- \sum_1^{\infty} A_n \sinh \frac{n\pi}{d} + \sum_1^{\infty} (-1)^n A_n \sinh \frac{n\pi}{d}}{- \sum_1^{\infty} A_n \frac{d}{n\pi} \left[1 - (-1)^n\right] \cosh \frac{n\pi}{d} + \sum_1^{\infty} A_n \left(\frac{d}{n\pi}\right)^3 \left[(n^2 \pi^2 - 2)(-1)^n + 2\right] \cosh \frac{n\pi}{d}} \quad \ldots\ldots \quad (71)$$

Although from Eq. 71, the existence of a solution cannot be guaranteed, the value $a\lambda = 0$ is excluded by consideration of the properties of eigenvalues. In fact, from the nature of the eigenvalue problem and the physics of the flow, there may be an infinite number of eigenvalues to satisfy Eq. 71. This result would be analogous to the infinite number of eigenvalues found by Curle in the unsteady jet problem. The smallest of these eigenvalues λ_{min}, will determine the slowest rate of decay. In itself, Eq. 71 does not determine $a\lambda$ because the A_n are unknown as yet and, moreover, will contain $a\lambda$. It constitutes a check on the computation of the A_n and this has been verified in a numerical example.

The boundary condition of Eq. 63 can be written as

$$\sum_1^\infty A_n \frac{n\pi}{d} \sinh \frac{n\pi}{d} \sin \frac{n\pi}{d}\alpha$$

$$= (a\lambda)^2 \left(1+\alpha^2\right) \sum_1^\infty A_n \cosh \frac{n\pi}{d} \sin \frac{n\pi}{d}\alpha$$

$$+ (a\lambda) \sum_1^\infty A_n \frac{n\pi}{d} \cosh \frac{n\pi}{d} \cos \frac{n\pi}{d}\alpha \quad \ldots\ldots\ldots\ldots (72)$$

Multiply each side by $\sin \frac{m\pi\alpha}{d}$ and integrate over the interval (o, d), to obtain

$$\frac{d}{2} A_m \frac{m\pi}{d} \sinh \frac{m\pi}{d}$$

$$= (a\lambda)^2 \sum_1^\infty \left[\int_o^d \left(1+\alpha^2\right) \sin \frac{m\pi\alpha}{d} \sin \frac{n\pi\alpha}{d}\, d\alpha\right] A_n \cosh \frac{n\pi}{d}$$

$$+ (a\lambda) \sum_1^\infty \left[\int_o^d \cos \frac{n\pi\alpha}{d} \sin \frac{m\pi\alpha}{d}\, d\alpha\right] A_n \frac{n\pi}{d} \cosh \frac{n\pi}{d} \quad \ldots\ldots (73)$$

Let

$$b_{m,n} = \int_o^d \left(1+\alpha^2\right) \sin \frac{m\pi\alpha}{d} \sin \frac{n\pi\alpha}{d}\, d\alpha,$$

$$c_{m,n} = \int_o^d \cos \frac{n\pi\alpha}{d} \sin \frac{m\pi\alpha}{d}\, d\alpha \quad \ldots\ldots (74)$$

Then, Eq. 73 can be written

$$\frac{d}{2} A_m \frac{m\pi}{d} \sinh \frac{m\pi}{d} = (a\lambda)^2 \sum_1^\infty A_n b_{m,n} \cosh \frac{n\pi}{d}$$

$$+ (a\lambda) \sum_1^\infty A_n \frac{n\pi}{d} c_{m,n} \cosh \frac{n\pi}{d} \quad \ldots\ldots\ldots\ldots (75)$$

Introduce a new coefficient

$$a_m = \frac{A_m}{2} m\pi \sinh \frac{m\pi}{d} \qquad (76)$$

and write Eq. 75

$$a_m = \sum_{n=1}^{\infty} a_n h_{m,n} (a\lambda) \quad (m = 1, 2, ...) \qquad (77)$$

in which

$$h_{m,n} (a\lambda) = (a\lambda)^2 \frac{2}{n\pi} b_{m,n} \coth \frac{n\pi}{d} + (a\lambda) \frac{2}{d} c_{m,n} \coth \frac{n\pi}{d} \qquad (78)$$

The integrations of Eq. 74 lead to

$$\left.\begin{aligned} b_{m,n} &= \frac{d^3}{\pi^2 (m-n)^2} \cos \pi (m-n) - \frac{d^3}{\pi^2 (m+n)^2} \cos \pi (m+n) \quad (m \neq n) \\ c_{m,n} &= -\frac{d}{2(m-n)\pi} [\cos \pi (m-n) - 1] - \frac{d}{2(m+n)\pi} [\cos \pi (m+n) - 1] \\ &\qquad (m \neq n) \\ b_{m,m} &= \frac{d}{2} + \frac{d^3}{6} - \frac{d^3}{4 m^2 \pi^2}, \quad c_{m,m} = 0 \end{aligned}\right\} \qquad (79)$$

Eqs. 77 represent a system of linear homogeneous equations, infinite in num ber and with an infinite number of unknowns a_1, a_2, a_3 . . . . This system ca be written explicitly as

$$\left.\begin{aligned} \left[h_{11} (a\lambda) - 1\right] a_1 + h_{12} (a\lambda) a_2 + h_{13} (a\lambda) a_3 + \ldots &= 0 \\ h_{21} (a\lambda) a_1 + \left[h_{22} (a\lambda) - 1\right] a_2 + h_{23} (a\lambda) a_3 + \ldots &= 0 \\ h_{31} (a\lambda) a_1 + h_{32} (a\lambda) a_2 + \left[h_{33} (a\lambda) - 1\right] a_3 + \ldots &= 0 \end{aligned}\right\} \qquad (80$$

For a nontrivial solution of this system of equations, its determinant shou vanish. Thus the concept of the infinite determinant as developed by Hill his Lunar theory appears in this analysis. Conditions for the convergence such a determinant are given by E. T. Whittaker and G. N. Watson.[11] Eqs. 8 are similar to Hill's equations except for the range of n in a_n. In Hill's cas n assumes the values n = . . . -2, -1, 0, 1, 2, . . . , while the parameter $a\lambda$ co responds to Hill's μ. Hill showed that, for the purpose of his astronomic problem, a good approximation to the value of μ could be obtained by consi ering only the three central rows and columns of the determinant. In a nume ical example, $a\lambda$ has been determined within 2% by considering only the smal est positive root of the equations $D_4 (a\lambda) = 0$, $D_6 (a\lambda) = 0$, $D_8 (a\lambda) = 0$, in whi the subscripts 4, 6, 8, denote the degree of the equations in $a\lambda$ which resu

11 "A Course of Modern Analysis," by E. T. Whittaker and G. N. Watson, The Ma millan Co., New York, 1947, pp. 36, 413.

from the successive consideration of the determinants two by two (rows and columns), three by three, and four by four, of the matrix

$$\begin{pmatrix} h_{11}-1 & h_{12} & h_{13} & h_{14} & \cdots \\ h_{21} & h_{22}-1 & h_{23} & h_{24} & \cdots \\ h_{31} & h_{32} & h_{33}-1 & h_{34} & \cdots \\ h_{41} & h_{42} & h_{43} & h_{44}-1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \quad \ldots\ldots\ldots \quad (81)$$

A good approximation of the value of $a\lambda$ is even determined by $D_2(a\lambda) = h_{11}(a\lambda) - 1 = 0$. Once the parameter $a\lambda$ is determined, one can compute the coefficients $h_{m,n}$ from Eqs. 78 and 79 and solve the system of Eq. 80 for a_1, a_2, a_3 . . . Next the coefficients A_n of Eq. 70 are computed by Eq. 76 and with this the problem is solved. In particular, the mode of motion of the unsteady free surface can be derived from Eq. 60; Thus

$$\delta_1 = \frac{1}{2}\,\frac{K}{\epsilon}\,\frac{q_o}{Q_{Ho}}\,\frac{\partial\phi_1}{\partial\beta} = \frac{1}{\lambda a}\sin\tau_o\,\frac{\partial\phi_1}{\partial\beta} \quad \ldots\ldots\ldots\ldots \quad (82)$$

because

$$q_o = -K\sin\tau_o \quad (27)$$

Free boundary condition Eq. 66: This boundary condition can be written as

$$\sum_1^{\infty} A_n \frac{n\pi}{d}\sinh\frac{n\pi}{d}\sin\frac{n\pi}{d}\alpha$$

$$= (a\lambda)^2(1+\alpha^2)\sum_1^{\infty} A_n \cosh\frac{n\pi}{d}\sin\frac{n\pi}{d}\alpha$$

$$+ 2(a\lambda)\sum_1^{\infty} A_n \frac{n\pi}{d}\cosh\frac{n\pi}{d}\cos\frac{n\pi}{d}\alpha$$

$$+ (a\lambda)\frac{\alpha}{1+\alpha^2}\sum_1^{\infty} A_n \cosh\frac{n\pi}{d}\sin\frac{n\pi}{d}\alpha$$

$$- \frac{\alpha}{(1+\alpha^2)^2}\sum_1^{\infty} A_n \frac{n\pi}{d}\cosh\frac{n\pi}{d}\cos\frac{n\pi}{d}\alpha$$

$$- \frac{1}{1+\alpha^2}\sum_1^{\infty} A_n \left(\frac{n\pi}{d}\right)^2\cosh\frac{n\pi}{d}\sin\frac{n\pi}{d}\alpha \quad \ldots\ldots\ldots \quad (83)$$

Multiply each side by $\sin \frac{m\pi\alpha}{d}$ and integrate over (o, d) to obtain

$$\frac{d}{2} A_m \frac{m\pi}{d} \sinh \frac{m\pi}{d}$$

$$= (a\lambda)^2 \sum_1^\infty \left[\int_o^d \left(1+\alpha^2\right) \sin \frac{m\pi\alpha}{d} \sin \frac{n\pi\alpha}{d} \, d\alpha \right] A_n \cosh \frac{n\pi}{d}$$

$$+ 2\,(a\lambda) \sum_1^\infty \left[\int_o^d \cos \frac{n\pi}{d}\alpha \sin \frac{m\pi}{d}\alpha \, d\alpha \right] A_n \frac{n\pi}{d} \cosh \frac{n\pi}{d}$$

$$+ (a\lambda) \sum_1^\infty \left[\int_o^d \frac{\alpha}{1+\alpha^2} \sin \frac{n\pi}{d}\alpha \sin \frac{m\pi}{d}\alpha \, d\alpha \right] A_n \cosh \frac{n\pi}{d}$$

$$- \sum_1^\infty \left[\int_o^d \alpha\left(1+\alpha^2\right)^{-2} \cos \frac{n\pi}{d}\alpha \sin \frac{m\pi\alpha}{d} \, d\alpha \right] A_n \frac{n\pi}{d} \cosh \frac{n\pi}{d}$$

$$- \sum_1^\infty \left[\int_o^d \frac{1}{1+\alpha^2} \sin \frac{n\pi}{d}\alpha \sin \frac{m\pi}{d}\alpha \, d\alpha \right] A_n \left(\frac{n\pi}{d}\right)^2 \cosh \frac{n\pi}{d} \quad \ldots\ldots\ldots (84)$$

Let

$$b_{m,n} = \int_o^d \left(1+\alpha^2\right) \sin \frac{m\pi\alpha}{d} \sin \frac{n\pi\alpha}{d} \, d\alpha; \quad c_{m,n} = \int_o^d \cos \frac{n\pi\alpha}{d} \sin \frac{m\pi\alpha}{d} \, d\alpha$$

$$d_{m,n} = \int_o^d \frac{\alpha}{1+\alpha^2} \sin \frac{m\pi\alpha}{d} \sin \frac{n\pi\alpha}{d} \, d\alpha;$$

$$\ldots\ldots (85)$$

$$e_{m,n} = \int_o^d \frac{\alpha}{\left(1+\alpha^2\right)^2} \cos \frac{n\pi\alpha}{d} \sin \frac{m\pi\alpha}{d} \, d\alpha$$

$$g_{m,n} = \int_o^d \frac{1}{1+\alpha^2} \sin \frac{m\pi\alpha}{d} \sin \frac{n\pi\alpha}{d} \, d\alpha$$

Then Eq. 84 can be written

$$\frac{d}{2} A_m \frac{m\pi}{d} \sinh \frac{m\pi}{d} = (a\lambda)^2 \sum_1^\infty A_n b_{m,n} \cosh \frac{n\pi}{d}$$

$$+ 2\,(a\lambda) \sum_1^\infty A_n \frac{n\pi}{d} c_{m,n} \cosh \frac{n\pi}{d} + (a\lambda) \sum_1^\infty A_n d_{m,n} \cosh \frac{n\pi}{d}$$

$$- \sum_1^\infty A_n e_{m,n} \frac{n\pi}{d} \cosh \frac{n\pi}{d} - \sum_1^\infty A_n g_{m,n} \left(\frac{n\pi}{d}\right)^2 \cosh \frac{n\pi}{d} \quad \ldots\ldots\ldots\ldots (86)$$

Now introduce a new coefficient

$$a_m = \frac{A_m}{2}\, m\pi \sinh \frac{m\pi}{d} \quad \ldots\ldots\ldots\ldots\ldots \quad (87)$$

and write Eq. 86 as

$$a_m = \sum_{n=1}^{\infty} a_n\, h_{m,n}(a\lambda) \quad (m = 1, 2, \ldots) \ \ldots \quad (88)$$

in which

$$h_{m,n}(a\lambda) = (a\lambda)^2 \frac{2}{n\pi} b_{m,n} \coth \frac{n\pi}{d} + (a\lambda)\frac{4}{d} c_{m,n} \coth \frac{n\pi}{d}$$

$$+ (a\lambda)\frac{2}{n\pi} d_{m,n} \coth \frac{n\pi}{d} - e_{m,n}\frac{2}{d} \coth \frac{n\pi}{d}$$

$$- g_{m,n}\frac{2}{d^2}\, n\pi \coth \frac{n\pi}{d} \quad \ldots\ldots\ldots\ldots\ldots \quad (89)$$

The remainder of the analysis is identical to that of the preceding section "Free boundary condition" except for the coefficients $d_{m,n}$, $e_{m,n}$, and $q_{m,n}$, which must be computed numerically.

TABLE 1.—$\Delta_1(a\lambda) = 0$ EQ. 90

$$\Delta_1(a\lambda) = \begin{vmatrix} 4.350(a\lambda)^2 - 1 & -0.793(a\lambda)^2 - 0.438(a\lambda) & 0.110(a\lambda)^2 & -0.031(a\lambda)^2 - 0.085(a\lambda) & \ldots \\ -1.975(a\lambda)^2 + 1.090(a\lambda) & 1.910(a\lambda)^2 - 1 & -0.557(a\lambda)^2 - 0.510(a\lambda) & 0.0964(a\lambda)^2 & \ldots \\ 0.417(a\lambda)^2 & -0.858(a\lambda)^2 + 0.787(a\lambda) & 1.260(a\lambda)^2 - 1 & -0.425(a\lambda)^2 - 0.545(a\lambda) & \ldots \\ -0.158(a\lambda)^2 + 0.435(a\lambda) & 0.199(a\lambda)^2 & -0.570(a\lambda)^2 - 0.548(a\lambda) & 0.947(a\lambda)^2 - 1 & \ldots \end{vmatrix} = 0$$

Numerical Examples.

Case of free boundary condition Eq. 65:

The mode of motion of the free surface and the exponential law have been computed for the family of dams for which[12]

$$d = \kappa \frac{K\,H}{Q_{Ho}} = 3$$

[12] "Unsteady-State Phenomena in the Flow Through Porous Media," by R. De Wiest, Tech. Report No. 3, Civ. Engrg. Dept., Stanford Univ., 1959, pp. 54–68.

These computations can be simply repeated for any other numerical value of d corresponding to the particular dam that must be investigated. Eqs. 73 and 78 have been computed numerically and lead to the value of

$$\Delta_1 (a\lambda) = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (90)$$

given in Table 1. The smallest positive roots of the successive determinants are

Determinant	$D_2(a\lambda)$	$D_4(a\lambda)$	$D_6(a\lambda)$	$D_8(a\lambda)$	
$a\lambda$	0.479	0.492	0.480	0.490	..(91)

Fig. 5 shows that there are no smaller roots than the afore-mentioned. From this, one may assume that the smallest root of $D_{2n}(a\lambda) = 0$ will converge to the smallest root of $\Delta_1(a\lambda) = 0$, as $n \rightarrow \infty$. The values of a_1, a_2, a_3, a_4, . . computed from the system of Eq. 77 will differ if one considers two, three, four, . . . equations and the corresponding value of $a\lambda$ which makes their determinant vanish, because these values of $a\lambda$ vary slightly. However, it is found that the values of the a_n decrease rapidly, so that it is sufficient to consider only the coefficients A_1 and A_2 in Eq. 70, for the value of $a\lambda = 0.49$ and $a_2 = 0.08\, a_1$. These computations and those of the mode of motion of the free surface are tabulated in Table 2. The mode of the free surface motion is represented in Fig. 6.

Appendix I shows that the continuity condition, Eq. 71, is satisfied for $a\lambda$ = 0.49. For an easier understanding of Table 2, recall that d = 3 and that β = at the free surface. Then

$$\phi_1 = A_1 \cosh\frac{\pi}{3} \sin\frac{\pi\alpha}{3} + A_2 \cosh\frac{2\pi}{3} \sin\frac{2\pi}{3}\alpha + \ldots.$$

$$= 2\frac{a_1}{\pi} \coth\frac{\pi}{3} \sin\frac{\pi\alpha}{3} + \frac{2}{\pi}\frac{a_2}{2} \coth\frac{2\pi}{3} \sin\frac{2\pi\alpha}{3} + \ldots \quad \ldots\ldots \quad (92)$$

Solution of the system Eq. 77 for a_1, a_2, a_3, a_4 gives

$$a_1 = 16.170\ c$$

$$a_2 = 1.295\ c \qquad \text{or} \qquad a_2 = 0.08\, a_1 \quad \ldots\ldots\ldots \quad (93)$$

$$a_3, a_4 \ll$$

The parameter c considers the homogeneity of the system, Eq. 77. Substitution of Eq. 93 into Eq. 92 and replacement of $\frac{2}{\pi}$ c by C* lead to

$$\phi_1 = C^*\left(20.60 \sin\frac{\pi\alpha}{3} + 0.67 \sin\frac{2\pi\alpha}{3} + \ldots\right) \quad \ldots \quad (9$$

and

$$\frac{\partial\phi_1}{\partial\alpha} = C^*\left(21.60 \cos\frac{\pi\alpha}{3} + 1.40 \cos\frac{2\pi\alpha}{3} + \ldots\right) \quad \ldots \quad (9$$

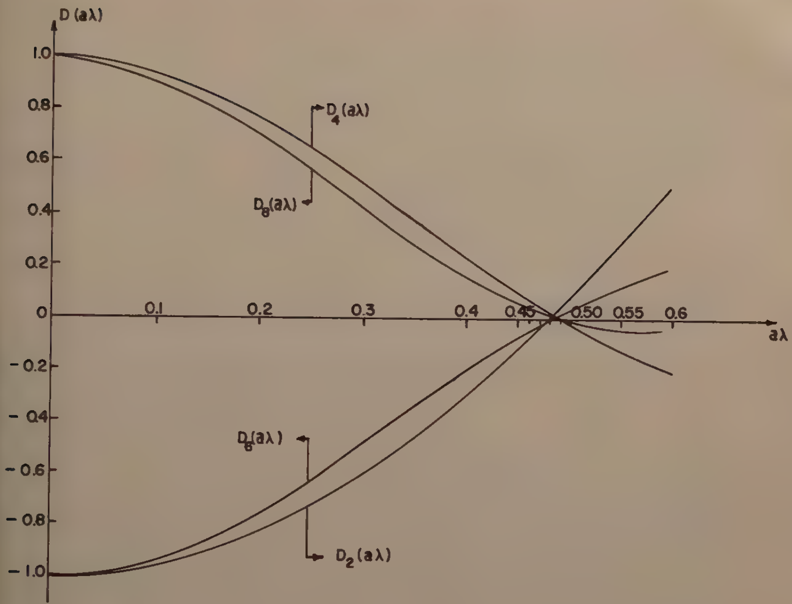

FIG. 5.—DETERMINATION OF a λ = 0.49

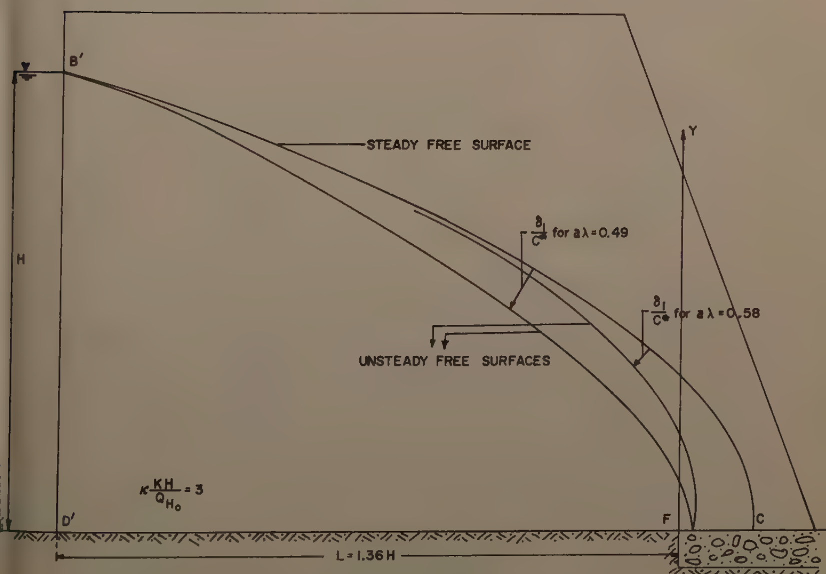

FIG. 6.—FREE SURFACES

TABLE 2.—FREE—SURFACE COMPUTATIONS

τ_o (1)	$\theta_o = 2\tau_o$ (2)	$\alpha = \cot g \tau_o$ (3)	$\sin \tau_o$ (4)	$\frac{\alpha\pi}{3}$ (5)	$\sin \frac{\alpha\pi}{3}$ (6)	$\cos \frac{\alpha\pi}{3}$ (7)	$\sin \frac{2\alpha\pi}{3}$ (8)	$\cos \frac{2\alpha\pi}{3}$ (9)	I (10)	II (11)	III (12)	IV (13)	Σ_{I}^{IV} (14)	$\frac{\delta_1}{C^*}$ (15)
20°	40	2.747	0.342	165	0.259	-0.966	-0.500	0.866	22.40	-1.40	-20.90	0.39	0.49	0.17
22.5	45	2.414	0.382	145	0.573	-0.819	-0.939	0.342	39.40	-2.10	-17.70	0.15	19.75	7.55
25	50	2.144	0.422	129	0.777	-0.629	-0.978	-0.208	44.10	-1.80	-13.60	-0.09	28.61	12.10
27.5	55	1.921	0.461	115.15	0.904	-0.426	-0.771	-0.636	42.90	-1.19	- 9.20	-0.29	32.22	14.80
30	60	1.732	0.500	104	0.970	-0.242	-0.469	-0.883	39.20	-0.62	- 5.23	-0.40	32.95	16.50
35	70	1.428	0.573	85.36	0.997	0.076	0.156	-0.987	30.60	0.15	1.64	-0.44	31.95	18.30
40	80	1.191	0.643	71.30	0.948	0.317	0.602	-0.798	23.20	0.48	6.85	-0.36	30.17	19.40
45	90	1.00	0.707	60	0.866	0.500	0.866	-0.500	17.50	0.57	10.80	-0.22	28.65	20.30
50	100	0.839	0.706	50.24	0.770	0.637	0.982	-0.187	13.20	0.55	13.80	-0.08	27.47	21.05
60	120	0.577	0.866	34.36	0.568	0.823	0.935	0.355	7.60	0.41	17.80	0.16	25.97	22.45
68	136	0.404	0.927	24.15	0.410	0.912	0.749	0.662	4.80	0.29	19.70	0.30	25.11	23.25
75	150	0.268	0.966	16.06	0.277	0.961	0.533	0.846	3.00	0.19	20.80	0.38	24.37	23.55
90	180	0	1.00	0	0	1	0	1	0	0	21.60	0.45	22.05	22.05

$I = 10.10(1 + \alpha^2) \sin \frac{\alpha\pi}{3}$ $\qquad$ $III = 21.60 \cos \frac{\pi\alpha}{3}$

$II = 0.33 \; (1 + \alpha^2) \sin \frac{2\pi\alpha}{3}$ $\qquad$ $IV = 0.45 \; \cos \frac{2\pi\alpha}{3}$

From Eq. 82 it follows that

$$\delta_1 = \sin \tau_o \left[a\lambda \left(1 + \alpha^2\right) \phi_1 + \frac{\partial \phi_1}{\partial \alpha} \right] \quad \ldots\ldots\ldots\ldots \quad (96)$$

and for $a\lambda = 0.49$, if one neglects the terms of smaller magnitude in the preceding series

$$\delta_1 = C^* \sin \tau_o \left[10.10 \left(1 + \alpha^2\right) \sin \frac{\pi \alpha}{3} \right.$$

$$\left. + 0.33 \left(1 + \alpha^2\right) \sin \frac{2 \pi \alpha}{3} + 21.60 \cos \frac{\pi \alpha}{3} + 0.45 \cos \frac{2 \pi \alpha}{3} \right] \quad \ldots \quad (97)$$

The value of C^* is arbitrary, but its choice determines the time origin of the unsteady phenomenon.

Case of free boundary condition Eq. 66:

TABLE 3.—COEFFICIENTS EQ. 85

$b_{m,n}$				
5.316	-2.431	0.513	-0.194	. . .
-2.431	5.828	-2.626	0.607	. . .
0.513	-2.626	5.923	-2.679	. . .
-0.194	0.607	-2.679	5.957	. . .
. . .	. . .	. . .	. . .	. . .
$c_{m,n}$				
0	-0.638	0	-0.129	. . .
1.269	0	-0.767	0	. . .
0	1.140	0	-0.824	. . .
0.502	0	1.084	0	. . .
. . .	. . .	. . .	. . .	. . .
$d_{m,n}$				
0.661	0.048	-0.047	-0.029	. . .
0.048	0.613	0.018	-0.068	. . .
-0.047	0.018	0.592	0.007	. . .
-0.029	-0.068	0.007	0.584	. . .
. . .	. . .	. . .	. . .	. . .
$e_{m,n}$				
0.095	-0.095	-0.073	-0.044	. . .
0.211	0.021	-0.139	-0.093	. . .
0.117	0.167	0.002	-0.149	. . .
0.071	0.097	0.157	-0.001	. . .
. . .	. . .	. . .	. . .	. . .
$g_{m,n}$				
0.533	0.256	0.080	0.031	. . .
0;256	0.614	0.287	0.090	. . .
0.080	0.287	0.623	0.291	. . .
0.031	0.090	0.291	0.624	. . .
. . .	. . .	. . .	. . .	. . .

TABLE 4.—$\Delta_2\ (a\lambda) = 0$ EQ.

$$
\Delta_2(a\lambda) = \begin{vmatrix}
4.350(a\lambda)^2 + 0.542(a\lambda) - 1.557 & -0.793(a\lambda)^2 - 0.860(a\lambda) - 0.304 & 0.110(a\lambda)^2 - 0.010(a\lambda) - 0.119 & -0.031(a\lambda)^2 - 0.177(a\lambda) - 0.059 & \ldots \\
-1.975(a\lambda)^2 + 2.219(a\lambda) - 0.408 & 1.910(a\lambda)^2 + 0.200(a\lambda) - 1.897 & -0.557(a\lambda)^2 - 1.020(a\lambda) - 0.510 & 0.096(a\lambda)^2\ 0.011(a\lambda) - 0.190 & \ldots \\
0.417(a\lambda)^2 - 0.038(a\lambda) - 0.171 & -0.860(a\lambda)^2 + 1.570(a\lambda) - 0.528 & 1.260(a\lambda)^2 + 0.125(a\lambda) - 2.310 & -0.426(a\lambda)^3 - 1.100(a\lambda) - 0.715 & \ldots \\
-0.158(a\lambda)^2 + 0.834(a\lambda) - 0.089 & 0.199(a\lambda)^2 - 0.022(a\lambda) - 0.233 & -0.570(a\lambda)^2 + 1.451(a\lambda) - 0.718 & 0.947(a\lambda)^2 + 0.092(a\lambda) - 2.745 & \ldots
\end{vmatrix} = 0
$$

The coefficients Eq. 85 have been evaluated on a computer, for the case = 3 and for a four by four matrix. The results are tabulated in Table 3. Eqs. 8 have been obtained numerically and lead to the value of

$$\Delta_2 (a\lambda) = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (98)$$

iven in Table 4. The smallest positive roots of the successive determinants re

Determinant	$D_2(a\lambda)$	$D_4(a\lambda)$	$D_6(a\lambda)$	$D_8(a\lambda)$
$a\lambda$	0.540	0.575	0.582	0.605

Fig. 7 shows that there are no smaller roots than the afore-mentioned. Again, one may assume that the smallest root of $D_{2n}(a\lambda) = 0$ will converge to the

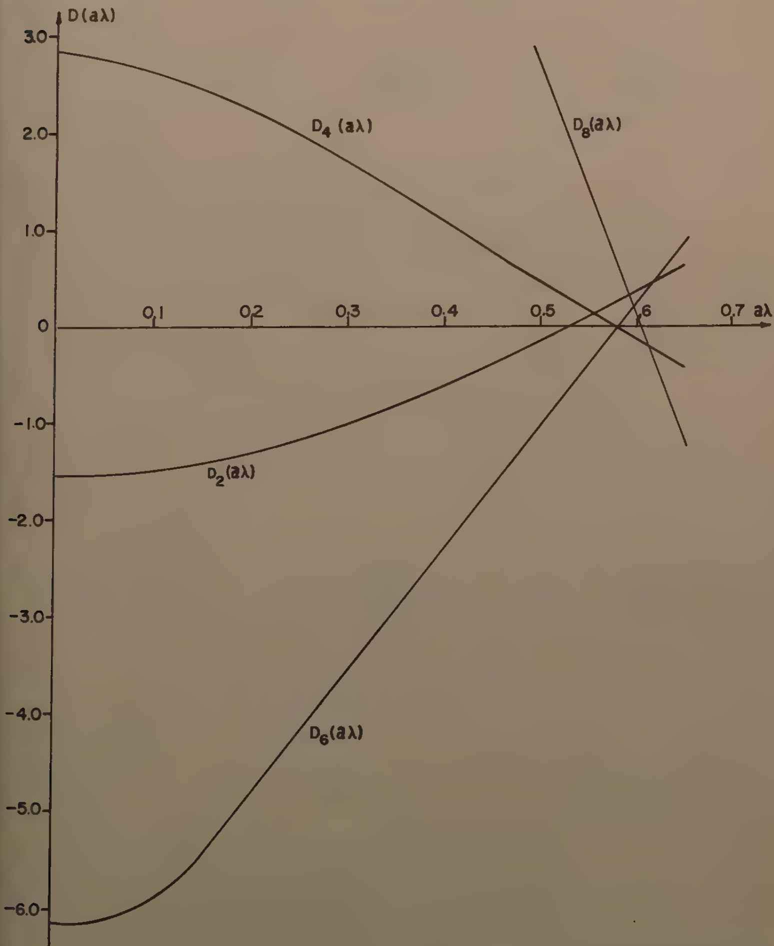

FIG. 7.—DETERMINATION OF a λ = 0.58

smallest root of $\Delta_2(a\lambda) = 0$ as $n \to \infty$, although the convergence seems to be slower than in the case of $\Delta_1(a\lambda)$. A value of $a\lambda = 0.58$ has been adopted to solve the homogeneous equations, Eqs. 88. The solution is

$$a_1 = 1.278\, c; \qquad a_2 = 0.267\, c; \qquad a_3 = -0.021\, c; \quad \ldots\ldots\ldots \quad (99)$$

with c a parameter. Substitution of Eq. 99 into Eq. 92 and replacement of $\frac{2}{10\pi}$ c by C* lead to

$$\phi_1 = C^* \left[16.4 \sin \frac{\pi\alpha}{3} + 1.38 \sin \frac{2\pi\alpha}{3} + \ldots\right] \quad \ldots\ldots (100)$$

$$\frac{\partial \phi_1}{\partial \alpha} = C^* \left[17.2 \cos \frac{\pi\alpha}{3} + 2.89 \cos \frac{2\pi\alpha}{3} + \ldots\right]$$

$$\frac{\partial^2 \phi_1}{\partial \alpha^2} = -C^* \left[18.0 \sin \frac{\pi\alpha}{3} + 6.05 \sin \frac{2\pi\alpha}{3} + \ldots\right]$$

Eq. 65 can be rewritten as

$$\frac{d\delta_1}{d\alpha} + (a\lambda)\left(1 + \alpha^2\right)\delta_1 = \left(1 + \alpha^2\right)^{1/2} \frac{\partial \phi_1}{\partial \beta} \quad \ldots \quad (101)$$

This is an ordinary differential equation of the form

$$\frac{d\delta_1}{d\alpha} + P(\alpha)\,\delta_1 = Q(\alpha) \quad \ldots\ldots\ldots\ldots\ldots \quad (102)$$

in which

$$Q(\alpha) = \left[(a\lambda)\,\alpha\left(1 + \alpha^2\right)^{-\frac{1}{2}} + (a\lambda)^2\left(1 + \alpha^2\right)^{3/2}\right]\phi_1$$

$$+\left[2(a\lambda)\left(1+\alpha^2\right)^{1/2} - \alpha\left(1+\alpha^2\right)^{-\frac{3}{2}}\right]\frac{\partial \phi_1}{\partial \alpha} + \left(1+\alpha^2\right)^{-\frac{1}{2}} \frac{\partial^2 \phi_1}{\partial \alpha^2} \quad \ldots\ldots \quad (103)$$

and is graphically represented in Fig. 8, and $P(\alpha) = (a\lambda)\left(1 + \alpha^2\right)$. The solution of Eq. 102 is

$$\delta_1 = e^{-\int_0^\alpha P\,d\alpha}\left[\rho + \int_0^\alpha Q\, e^{\int_0^\alpha P\,d\alpha}\, d\alpha\right] \quad \ldots\ldots \quad (104)$$

in which the integration constant ρ can be determined from the condition $\delta_1 =$ for $\alpha = 3$. It follows that

$$\rho = -\int_0^3 Q\, e^{\int_0^3 P\,d\alpha}\, d\alpha \quad \ldots\ldots\ldots\ldots\ldots \quad (105)$$

The integrations contained in Eq. 104 have been done simply by trapezia approximations. The results are summarized in Table 5.

The free surface mode is represented in Fig. 6. The value of C * is again arbitrary, determining the time origin of the unsteady phenomenon. It is obvious that the unsteady free surfaces of Fig. 6 are not occurring at the same

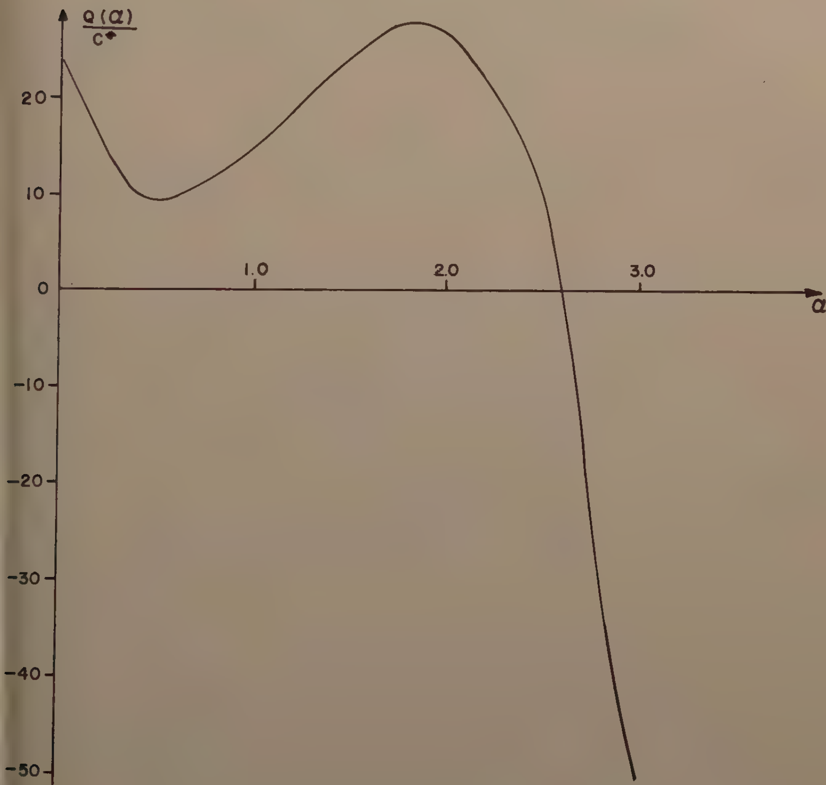

FIG. 8.—GRAPHICAL REPRESENTATION OF EQ. 93

me. In both cases it will take a different time interval from the time origin, reach the positions shown.

EXPERIMENTS

The Hele-Shaw model, or parallel plate model, has been used extensively to mulate potential flow of fluids since it was first introduced in 1897 and since e mathematical theory, proving its value, was developed by Stokes in 1899. consists of two closely-spaced parallel plates between which a viscous fluid ws. Horizontal models, and vertical, which are particularly well-suited for

TABLE 5.—FREE—SURFACE COMPUTATIONS

α (1)	$\int_0^{\alpha} P d\alpha$ (2)	$e^{\int_0^{\alpha} P d\alpha}$ (3)	$\frac{Q(\alpha)}{C^*}$ (4)	$\frac{1}{C^*}\int_0^{\alpha} Q e^{\int_0^{\alpha} P d\alpha} d\alpha$ (5)	$e^{-\int_0^{\alpha} P d\alpha}$ (6)	$\frac{1}{C^*}\left[e + \int_0^{\alpha} Q e^{\int_0^{a} P d\alpha} d\alpha\right]$ (7)	$\frac{\delta_1}{C^*}$ (8)	$\frac{\delta_1}{\delta_{1\,max}}$ (9)
0	0	1	23.30	0	1	6,407	6,407	1.000
0.268	0.161	1.175	12.75	5.13	0.85	6,412	5,460	0.850
0.404	0.249	1.283	9.30	6.96	0.778	6,414	4,980	0.775
0.577	0.374	1.454	9.30	9.17	0.688	6,416	4,420	0.688
0.839	0.604	1.830	12.30	13.89	0.547	6,421	3,510	0.546
1.00	0.777	2.175	15.43	18.40	0.460	6,425	2,960	0.462
1.191	1.021	2.780	19.27	26.70	0.359	6,434	2,305	0.359
1.428	1.394	4.030	23.69	44.40	0.248	6,451	1,600	0.249
1.732	2.014	7.48	27.20	89.90	0.134	6,497	870	0.136
1.921	2.491	12.15	27.22	140.40	0.082	6,547	537	0.084
2.144	3.153	23.50	23.87	239.90	0.042	6,647	282	0.044
2.414	4.121	61.80	13.19	425.90	0.016	6,834	110	0.017
2.747	5.511	250	-15.11	-67.10	0.004	6,340	25	0.004
3.00	6.801	905	-51.10	-6,407	0.001	0	0	0

ree surface flows, have been constructed. An excellent survey of the litera-
ure on the subject is given by D. K. Todd,[13] M. ASCE.

Design of Apparatus.—In the design of the apparatus, one must make com-
romises in some way when considering the proper dimensions of the model.
arge models are expensive, small models inaccurate because of increased
ffect of capillarity, small time-scale, difficult observations of displacements,
nd measurement of small discharges. Because the steady state discharges
nd effective drain-lengths for small $\frac{H_m}{L_m}$ should be approximately given by

$$Q_{H,\infty} = K_m\, b \left[\left(H_m^2 + L_m^2\right)^{1/2} - L_m\right] \quad \ldots\ldots\ldots (106)$$

nd

$$\frac{1}{2\,K_m}\, Q_{H,\infty} \quad \ldots\ldots\ldots\ldots\ldots (107)$$

fter Eq. 21 and the coordinate of C in Fig. 1(a), a lower limit on L_m can be
ound by estimating these values for $\frac{H_m}{L_m} = 0.1$. To find a suitable height for
e model, the model scales must be derived first. The equation of motion of
e free surface of the model flow is

$$\frac{\epsilon_m}{K_m}\, \frac{\partial h_m}{\partial t_m} = \frac{\partial \phi_m}{\partial x_m}\, \frac{\partial h_m}{\partial x_m} - \frac{\partial \phi_m}{\partial y_m} \quad \ldots\ldots\ldots\ldots (108)$$

ith

$$\phi_m = y_m + \frac{p_m}{\gamma_m} \quad \ldots\ldots\ldots\ldots\ldots\ldots (109)$$

hese equations are analogous to Eqs. 45 and 41.

$$\epsilon_m = 1$$

nd

$$K_m = \frac{1}{12}\, g\, \frac{b^2}{\nu_m} \quad \ldots\ldots\ldots\ldots\ldots (110)$$

which g is the gravitational acceleration, b the constant spacing of the plates,
nd ν_m the kinematic viscosity of the viscous liquid. The scales $\bar{u}_x$, $\bar{u}_y$, and
follow directly from compatibility relations between Eqs. 45 and 108.

13 "Unsteady Flow in Porous Media by Means of a Hele-Shaw Viscous Fluid Model," D. K. Todd, Transactions, Amer. Geophysical Union, Vol. 35, 1954, No. 6, p. 905.

Let

$$
\begin{aligned}
h_m &= h\,\bar{u}_y \\
t_m &= t\,\bar{u}_t \\
x_m &= x\,\bar{u}_x \\
y_m &= y\,\bar{u}_y \\
\phi_m &= \phi\,\bar{u}_\phi
\end{aligned}
\qquad \ldots\ldots\ldots\ldots\ldots \quad (111)
$$

Then

$$
\left.
\begin{aligned}
\frac{\partial h}{\partial t} &= \frac{\bar{u}_t}{\bar{u}_y}\,\frac{\partial h_m}{\partial t_m}\,; &\quad \frac{\partial \phi}{\partial x} &= \frac{\bar{u}_x}{\bar{u}_\phi}\,\frac{\partial \phi_m}{\partial x_m} \\
\frac{\partial h}{\partial x} &= \frac{\bar{u}_x}{\bar{u}_y}\,\frac{\partial h_m}{\partial x_m}\,; &\quad \frac{\partial \phi}{\partial y} &= \frac{\bar{u}_y}{\bar{u}_\phi}\,\frac{\partial \phi_m}{\partial y_m}
\end{aligned}
\right\}
\qquad \ldots\ldots\ldots\ldots \quad (112)
$$

Substitution of Eq. 112 into Eq. 45 leads to

$$
\frac{\epsilon}{K}\,\frac{\bar{u}_t}{\bar{u}_y}\,\frac{\partial h_m}{\partial t_m} = \frac{\bar{u}_x^2}{\bar{u}_\phi\,\bar{u}_y}\,\frac{\partial \phi_m}{\partial x_m}\,\frac{\partial h_m}{\partial x_m} - \frac{\bar{u}_y}{\bar{u}_\phi}\,\frac{\partial \phi_m}{\partial y_m} \qquad \ldots\ldots\ldots \quad (113)
$$

If both sides of this equation are multiplied by $\frac{K}{\epsilon}\,\frac{\bar{u}_y}{\bar{u}_t}\,\frac{\epsilon_m}{K_m}$, then

$$
\frac{\epsilon_m}{K_m}\,\frac{\partial h_m}{\partial t_m} = \frac{K}{\epsilon}\,\frac{\bar{u}_x^2}{\bar{u}_\phi\,\bar{u}_t}\,\frac{\epsilon_m}{K_m}\,\frac{\partial \phi_m}{\partial x_m}\,\frac{\partial h_m}{\partial x_m} - \frac{K}{\epsilon}\,\frac{\bar{u}_y^2}{\bar{u}_t\,\bar{u}_\phi}\,\frac{\epsilon_m}{K_m}\,\frac{\partial \phi_m}{\partial y_m} \quad \ldots \quad (114)
$$

From Eqs. 118 and 114, if one considers that $\bar{u}_\phi = \bar{u}_y$ from and that $\epsilon_m = 1$ the compatibility conditions are

$$
\frac{K}{\epsilon}\,\frac{\bar{u}_x^2}{\bar{u}_y\,\bar{u}_t}\,\frac{1}{K_m} = 1 \qquad \ldots\ldots\ldots\ldots\ldots \quad (115a)
$$

and

$$
\frac{K}{\epsilon}\,\frac{\bar{u}_y}{\bar{u}_t}\,\frac{1}{K_m} = 1 \qquad \ldots\ldots\ldots\ldots\ldots \quad (115b)
$$

From Eqs. 115 it follows that, in order to have a unique time scale, one must have $\bar{u}_x = \bar{u}_y$. This means that the model cannot be distorted, which is was, to our knowledge at least, in most of the previously constructed models of this kind. The distortion of the model is a direct consequence of the Dupuit assumption, which was not made in the present analysis. The opposite must be equally true: distorted models tend to justify Dupuit's assumptions. From Eq. 104 the time scale is

$$
\bar{u}_t = \frac{1}{\epsilon}\,\frac{K}{K_m}\,\bar{u}_x\,, \quad \text{with } \bar{u}_x = \bar{u}_y \qquad \ldots\ldots\ldots\ldots \quad (116)
$$

ϵ, K are given for each case under investigation. One disposes of $\bar{u}_x = \bar{u}_y$ and of K_m, through choice of the model and viscous liquid characteristics, to find a suitable time scale $\bar{u}_t$. To have a large time scale, a small K_m is desired, requiring close spacing and highly viscous liquid. To reduce the effect of capillarity to a negligible value, a spacing b = 4.84 mm has been adopted. A crystal white, chemically pure 96% glycerol aqueous solution served as viscous liquid. At 20° C, Eq. 110 gives K_m = 2.88 cm per sec. To fix the ideas, it is convenient to think of the model as representing a river levee at a length scale 1:10 or a large earth embankment, retaining a reservoir, at a length scale 1:100. Therefore a L_m = 132.2 cm has been adopted. The net maximum height H_m = 97.7 cm corresponds to $\frac{H_m}{L_m} = 0.74$. The corresponding estimates from Eqs. 106 and 107 are

$\frac{H_m}{L_m}$	$Q_{H,\infty}$	$\frac{1}{2K_m} Q_{H,\infty}$, in centimeters
0.1	0.92 cm^3/sec	0.33
0.74	45.01 cm^3/sec	16.13

These values are not difficult to measure.

Description of Apparatus.—Close-up photographs of the model are shown in Fig. 9 and Fig. 10. Two 1/2-in. thick plexiglass plates resting on a sump are kept at a constant distance of 0.484 cm by means of aluminum strips at top and bottom and by spacers whose interference in the flow picture is negligible. The aqueous glycerol solution is circulated in a closed cycle from the sump by a gear pump. It is pumped to a storage tank suspended at the ceiling of the air-conditioned room, and flows by gravity into the entrance reservoir attached to the plates. This flow is controlled by a fine regulating valve.

A movable overflow in the entrance reservoir is raised or lowered by a variable-speed motor capable of continuous speed control in a 2.3:1 range. Limit switches are built in for $\frac{H_m}{L_m}$ = 0.1 and 0.74. With the fine regulating valve, the flow into the entrance reservoir can be easily adjusted so that the level in this reservoir rises at the same speed as the funnel of the overflow. The discharge through the plates is drained into a tank. This tank is supplied with a variable overflow which can be adjusted accordingly to the discharge through the plates, so that the boundary conditions of the underdrain can be exactly simulated at all times. From the drainage tank, the liquid flows back to the sump. Flow rates can be computed by weighing quantities of liquid collected during known time intervals and by simultaneous specific weight determinations. Dye can be injected between the plates at different levels of the entrance reservoir and at the free surface, to visualize the streamlines.

The model was first calibrated by twenty steady state flows at different $\frac{H_m}{L_m}$ values and in a temperature range from 20°C to 26°C. All measured $Q_{H,m}$ values were then reduced to 20° C, assuming that $Q_{H,m}$ was linearly related to K_m. The K_m values were reduced to 20° C by making simultaneous measurements of ν_m by means of an Ostwald viscosimeter. Although tiny air bub-

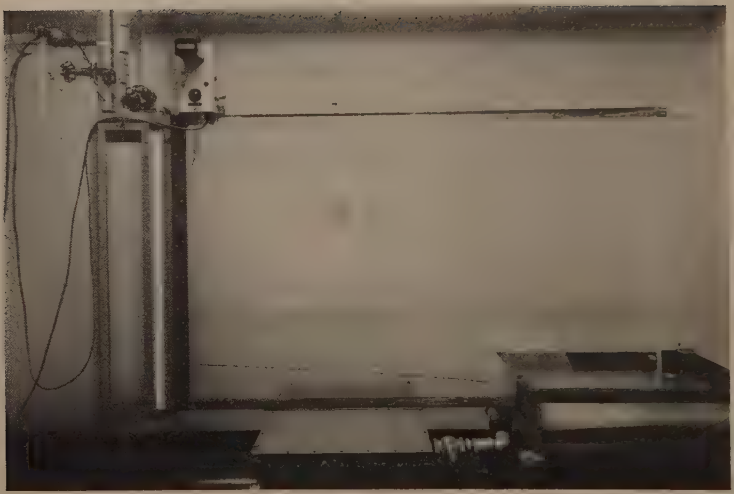

FIG. 9.—STEADY MODEL FLOW FOR $\frac{H_m}{L_m}$ = 0.74

FIG. 10.—STEADY MODEL FLOW FOR $\frac{H_m}{L_m}$ = 0.1

bles developed in the glycerol solution during the tests because of the pumping, specific gravity measurements with a Westphal balance showed only variations from 1.259 to 1.252 over the period of testing. The influence of these bubbles on ν_m was also negligible, as the variation of ν_m with temperature closely followed the curve for the 96% glycerol aqueous solution. The emulsion moreover added to the visualization of the flow (Figs. 9 and 10). These figures show the deviation of the free surface computed from Eq. 20 and the one that develops in the model for $\frac{H_m}{L_m} = 0.74$ and 0.1. The computed free surfaces are drawn in pencil on the plates. Fig. 9 also shows two intermediate positions of the un-

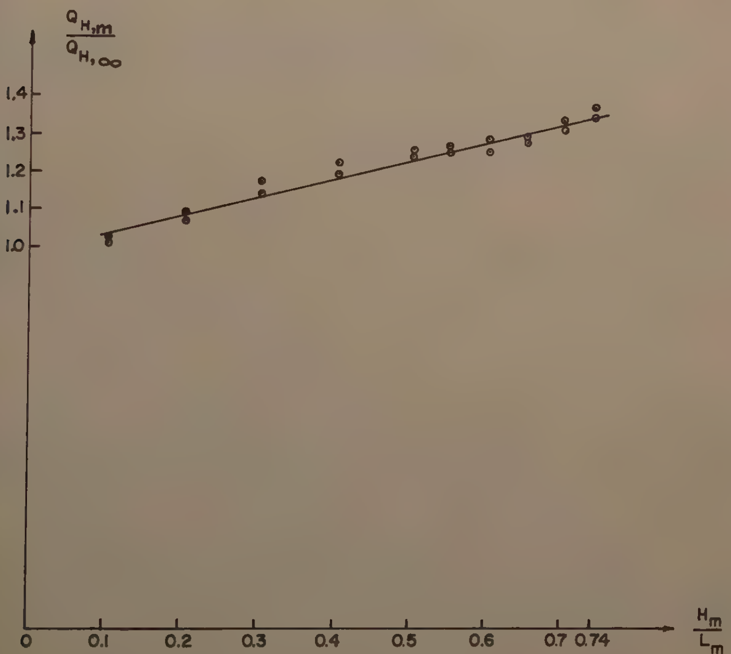

FIG. 11.—EFFECT OF $\frac{H_m}{L_m}$ ON DISCHARGE

teady free surface, the upper one for $a_m \lambda_m = 0.58$, the lower one for $a_m \lambda_m$ 0.49. Fig. 11 shows how the measured values $Q_{H,m}$ tend to deviate from $Q_{H,\infty}$.

Hypothesis and Testing Procedure.—The hypothesis consists of the assumption that the time origin of the exponential law $e^{-\lambda t}$ coincides with the moment at the head behind the dam reaches its final value after being raised from a w value. This hypothesis is to be tested against the behavior of the flow in e model. If the hypothesis is correct, then $\delta_{1,m}$ can be observed and it is ossible to compute and to observe the time interval during which $\delta_{1,m}$ de-

creases to $0.01\ \delta_{1,m}$. In the case of a rapid rise, the level in the entrance reservoir is raised from $0.1\ L_m$ to $0.74\ L_m$ at the approximately constant speed of 2.30 cm per sec. When $H_m = 0.74\ L_m$ is reached, $\delta_{1,m}$ is observed and then the time $t_{m,c}$ is measured during which $\delta_{1,m}$ decreases to $0.01\ \delta_{1,m}$. This run is easily reproduced and the average of five consistent results is given. For each run $Q_{H,m}$ is measured, ν_m is read from the curve at the measured temperature, and K_m is computed. Then $a_m = \dfrac{\epsilon_m Q_{H,m}}{K^2_m}$, and $\lambda_m = 0.58/a_m$. From $e^{-\lambda_m t_{c,c}} = 0.01$, the computed time $t_{c,c}$ follows and is compared with $t_{m,c}$. In the case of a slow rise, the nearly constant speed at which the level in the entrance reservoir rises is 1 cm per sec, and $\lambda_m = 0.49/a_m$. Intermediate positions of the unsteady free surface were drawn on the plates, such as those for which $\delta_{1,m}$ was reduced to $0.5\ \delta_{1,m}$. Computed values of the time interval $t_{c,0.5}$ required for this reduction and observed times $t_{m,0.5}$ were compared.

Summary of Results.—

Rises from $0.1\ L_m$ to $0.74\ L_m$. (Five runs)

Fast	Slow
Speed: 2.30 cm/sec	Speed: 1 cm/sec
$t_{m,c} = 44.5$ sec; $\delta_{1,m} = 12.2$ cm	$t_{m,c} = 52.0$ sec; $\delta_{1,m} = 6.2$ cm
$t_{c,c} = 48.5$ sec	$t_{c,c} = 60.5$ sec
	$t_{m,0.5} = 7.1$ sec
	$t_{c,0.5} = 8.1$ sec

No consistent results were obtained for the 50% decay in five runs of fast rise, because of the difficulty of observation. Recording with a movie camera would be necessary in this case.

Interpretation of Results.—There is a strong indication that the hypothesis concerning the time origin is true. This was by no means obvious before the tests. Computed and observed values are of the same order of magnitude and rather close. The trend of the observed values to be smaller than the computed was general.

The most difficult element to measure was the temperature in the liquid. This measurement was made in the drain. It was difficult to keep a constant temperature during a complete run. Although all measurements were reduced to 20° C, the possibility of adding small errors existed. Cooling of the liquid before flow into the entrance reservoir would be necessary if more accuracy is desired.

The intermediate positions of the unsteady free surface were drawn on the plates from the actual steady free surface in the model. Because a_m, and hence λ_m, in the model changes with temperature, it was difficult to obtain good agreement over the whole length of the free surface, because the temperature of the liquid in the model was not sufficiently uniform. However, the shape of the front in the vicinity of the underdrain was steeper in the case of the rapid rise than in the case of the slow rise, as predicted by the theory.

The value of 1 sec in the model can be illustrated by an example.

Given: River levee at $\bar{u}_x = \bar{u}_y = 1{:}10$ $\epsilon = 0.35$; $K = 2 \times 10^{-3}$ cm per sec; $K_m = 2.88$ cm per sec; and $\bar{u}_t = 1.985 \times 10^{-4}$.

One sec in the model corresponds to 1 hr 23 min 50 sec in the prototype. A fast rise in the model of 37 sec corresponds to a rise of the river level of (0.74 - 0.1) 13.22 m = 8.45 m in 51.8 hr. The same test in the model also represents the filling of the reservoir behind a large dam $\left(\bar{u}_x = \bar{u}_y = \frac{1}{75}\right)$ in which the water level rises 63.4 m in 16.25 days. Both cases may be considered as rapid water rises. A slow rise in the model of 1 ft 25 in. corresponds to a rise of 8.45 m of the river level behind the previously mentioned levee in 5 days.

CONCLUSIONS

1. It follows both from theory and experiments that the damping of the unsteady motion under consideration is faster for a fast level rise than for a slow rise. Both analysis and experiment, however, show that the difference is not important and for engineering purposes the simpler analysis of boundary condition Eq. 63, leading to a λ = 0.49, is satisfactory.

2. The unsteady free surface never overshoots the final steady surface, not even in the case of rapid rises. Negative values of δ_1, although possible in theory, never occurred in the analysis and were never observed in the model.

3. In most soils, where Darcy's law is valid, K is nearly proportional to ϵ^2. In first approximation then, a is inversely proportional to ϵ and proportional to $\left[(H^2 + L^2)^{1/2} - L\right]$. The damping is faster for small values of a, hence for porous dams and long dams.

4. In the case where the level is raised from 0.1 L to 0.74 L, it follows from the analysis that $d = \kappa \frac{K\ H}{Q_{Ho}}$, and for d = 3, κ = 0.945, $\frac{K\ H}{Q_{Ho}}$ = 3.18, so that $a = 0.315 \frac{\epsilon H}{K}$. Assuming $a\lambda$ = 0.49 for all cases, then $\lambda = 1.555 \frac{K}{\epsilon H}$. The time t_c that will elapse before the unsteady motion is damped out to 1% of its original value after the head has reached its maximum value H is then given by $t_c = 1.285\ \epsilon \frac{H}{K}$.

5. Further model experiments can determine the range of validity of the analytical results, insofar as speed of rise and amplitude of rise are concerned. Intuitively one feels that these elements can be made so small that the unsteady state phenomena can be treated as a continuous succession of steady-state flows. For smaller amplitudes of the rising level, the spacing of the plates would have to be reduced.

6. For another case of unsteady free surface flow, where the analysis was left with an undetermined parameter c, the reader is referred to Arthur Casagrande, F. ASCE, and W. L. Shannon,[14] F. ASCE. This parameter is also determined by a Hele-Shaw model, by comparing the time factor for 50% drainage with the actual time required for drainage in the model.

7. The Hele-Shaw model can be used to simulate the flow through geometrically more complicated types of dams than the one considered in this paper. To interpret the results correctly, care should be given to proper scaling of

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14 "Base Course Drainage for Airport Pavements," by A. Casagrande and W. L. Shannon, Transactions, ASCE, Vol. 117, 1952, p. 792.

the model, possibly by a previous analysis of that part of the flow picture that lends itself to this purpose.

ACKNOWLEDGMENTS

This paper consists of an analytical extension of part of a Ph. D. dissertation written at the Civil Engineering Department of Stanford University Stanford, Calif., under NSF grant G-4126 and of experimental work supported by research funds allocated by a group of firms to Princeton University, Princeton, N. J. The Hele-Shaw model was designed by the writer and built in the Hayes Engineering Shop of Princeton University.

APPENDIX I.—CONTINUITY CONDITION

Consider Eq. 71

$$(a\lambda)^2 = \frac{-\sum_1^\infty A_n \sinh\frac{n\pi}{d} + \sum_1^\infty (-1)^n A_n \sinh\frac{n\pi}{d}}{-\sum_1^\infty A_n \frac{d}{n\pi}\left[1-(-1)^n\right]\cosh\frac{n\pi}{d} + \sum_1^\infty A_n\left(\frac{d}{n\pi}\right)^3\left[\left(n^2\pi^2-2\right)(-1)^n+2\right]\cosh\frac{n\pi}{d}} \quad \ldots \quad (71)$$

or

$$(a\lambda)^2 = \frac{N}{D} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (117)$$

Since

$$N = -2A_1 \sinh\frac{\pi}{d} = -2\,\frac{a_1}{\pi} = -\frac{2}{\pi} \times 16.17\ c = -10.28c \quad \ldots\ldots\ldots \quad (118)$$

and

$$D = -A_1\frac{d}{\pi}\,2\cosh\frac{\pi}{d} + A_1\left(\frac{d}{\pi}\right)^3\left(-\pi^2+4\right)\cosh\frac{\pi}{d} + A_2\left(\frac{d}{2\pi}\right)^3 4\pi^2\cosh\frac{2\pi}{d} \quad \ldots\ldots\ldots\ldots \quad (119)$$

$$= - \frac{2a_1}{\pi}\,\frac{3}{\pi}\,2\coth\frac{\pi}{3} + \frac{2a_1}{\pi}\left(\frac{3}{\pi}\right)^3\left(-\pi^2+4\right)\coth\frac{\pi}{3}$$

$$+ \frac{2a_2}{2\pi}\left(\frac{3}{2\pi}\right)^3 4\,\pi^2\coth\frac{2\pi}{3}$$

$$= - 10.28c\left(\frac{3}{\pi}\right) 2\ (1.282) + 10.28c\left(\frac{3}{\pi}\right)^3\left(-\pi^2+4\right) 1.282$$

$$+ 0.412c\left(\frac{3}{2\pi}\right)^3 4\,\pi^2\ (1.032)$$

$$= - 25.2c - 67.2c + 2.90c = - 89.5c$$

Then

$$(a\lambda)^2 = \frac{10.28c}{89.50c} = 0.23$$

and

$$a\lambda = 0.48\ ,$$

which is sufficiently close to $\lambda = 0.49$.

APPENDIX II.—NOTATION

The following symbols, adopted for use in this paper, conform essentially with "American Standard Letter Symbols for Hydraulics" (ASA Z10.2 -- 1942), prepared by a committee of the American Standards Association with Society representation, and approved by the Association in 1942:

$a = \frac{\epsilon Q_{Ho}}{K^2}$ = grouped with λ to form dimensionless parameter $(a\lambda)$;

b = spacing of plates

$d = \kappa \frac{K\,H}{Q_{Ho}}$ = dimensionless constant in (α, β) plane to characterize dam of finite length;

H = constant head;

K = hydraulic conductivity;

k = permeability of porous medium;

L = length of finite dam;

p = pressure in water;

q = local velocity;

Q_{Ho} = steady seepage rate per unit width of dam for head H;

$Q_{H,\infty}$ = flow rate in model computed with formula for semi-infinite dam;

$Q_{H,m}$ = measured flow rate in model;

s = length along free surface;

t = time;

$t_{c,c}$ • = computed time for 0.99 decay, based on model characteristics;

$t_{m,c}$ = observed time for 0.99 decay;

$t_{c,0.5}$ = computed time for 0.50 decay; based on model characteristics;

$t_{m,0.5}$ = observed time for 0.50 decay;

u, v = components of q;

$w = u + iv$ = complex velocity;

W = complex potential;

$\alpha = \frac{Ku_o}{q_o^2}$; $\beta = -\frac{Kv_o}{q_o^2}$ = dimensionless hodograph coordinates;

γ = unit weight of water;

δ_1 = normal displacement;

ϵ = porosity of medium;

κ = dimensionless constant depending on $\frac{H}{L}$ ratio of dam;

λ = eigenvalue;

ν_m = kinematic viscosity of liquid

τ_o = $\tan^{-1} \frac{v_o}{u_o}$;

Ψ = stream function;

ϕ = hydraulic head; potential;

ϕ_o = steady potential;

ϕ_1 = perturbation potential; and

Φ = potential.

The subscript m is used to define elements or characteristics of the model.

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DISCUSSION

A DISCHARGE FORMULA FOR FLOW IN STRAIGHT ALLUVIAL CHANNELS[a]

Closure by Shoi-Yan Hwang

SHOI-YAN HWANG,[1] A. M. ASCE.—The writers are indebted to the discussers who, by contributing analyses which re-examine some of the ideas in the paper, have added greatly to its value.

Blench first asked the question about the scale effect on the prototype from the laboratory system. As mentioned in the paper, the discharge coefficient probably has to be modified to suit the field condition. Results obtained from the laboratory data usually offer a guide to the solution of the field problem. Analysis of the available field data along the writers' suggested direction could probably lead to a new set of empirical curves for natural canals.

Blench also questioned the effects of the non-uniformity of the bed materials on the resistance to flow. This problem is extremely complex and still remains unsolved. It is thus customary to use a representative size for the grain mixtures.

As to the effects of suspended sediment load on the resistance, Vito A. Vanoni, F. ASCE, showed[2] that its effect is not as significant as the other factors such as the sediment properties and the flow characteristics. D. B. Simons, M. ASCE, et al., has shown[3] that large concentrations of clay can reduce the effective size of the bed material by reducing its fall velocity and hence affect the resistance to flow. However, all the indices and curves as presented were from data having sediment load. The energy spent on moving particles should have simultaneously affected the other variables such as depth or velocity of flow, that is, the effects of charge on the boundary resistance have been taken into consideration implicitly. Blench seems to favor placing Manning's 2/3 by 3/4 for movable bed of natural canals. This problem is beyond the scope of the writers' paper. Manning's formula stemmed from the empirical point of view and has been particularly applicable to a fixed rough boundary. Modifications of its exponent from 2/3 to 3/4 for movable beds of natural canals may be plausible although this has not yet been clearly demonstrated with many data taken over a wide range of conditions.

Garde, has extended the writers' work to the natural canals. The new criterion for determining the boundary regime, which thus leads to a modified C_d:d curve for natural canals, is invaluable. Garde's work confirmed that the writers' discharge formula originally for straight laboratory alluvial channels

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[a] November 1959, by H. K. Liu and S. Y. Hwang (Proc. Paper 2260).

[1] Graduate Student, Dept. of Engrg. Mechanics, Univ. of Kansas, Lawrence, Kans.

[2] "Resistance Properties of Sediment-Laden Streams," by Vito A. Vanoni and George N. Nomicos, Proceedings, ASCE, Vol. 85, No. HY 5, May, 1959.

[3] "Flume Studies Using Medium Sand and Bentonite," by D. B. Simons, E. V. Richardson, and W. H. Haushild, CSU Report CER60DBS10, 1960.

is of a correct form and is applicable to the field condition with modification. More field data should be collected and analyzed according to Garde's work so that discharge in natural canals can be predicted with more accuracy.

Mr. Bogardi questioned the ambiguity of the invariant group K. The writer admits that the combination of the individual parameters which form K cannot be interpreted physically. However, by properly selecting the exponents m, λ, n in the invariant group K and using the relation

$$W = \sqrt{\frac{4}{3}\,\frac{1}{C_d}\,\frac{\rho s-\rho}{\rho}}$$

for the fall velocity, Eq. 24 can be reduced to

$$\frac{V}{V_*} = A_1 \left(\frac{V_* R}{\nu}\right)^{1/7} \qquad \text{(A)}$$

for fixed smooth boundary, and

$$\frac{V}{V_*} = A_2 \left(\frac{R}{d}\right)^{1/6} \qquad \text{(B)}$$

for fixed rough boundary.

Eqs. A and B are of the exponential forms of velocity distribution after G. H. Keulegan.[4] This indicates that the invariant group K does conform with the existing knowledge of boundary resistance.

Mr. Lean questioned the general applicability of the index K and the coefficient C_a. Since most of the data were drawn from the laboratory and correlated empirically, it does not guarantee that the resulting relationships are applicable to the condition which is more uncontrolled. Garde has used the fielc data to examine Eq. 1 and the C_a:d - curve has been modified to fit the fielc condition.

The writer is not clear as to the meaning of the statement "the evidence tha for a ripple bed the velocity for a given depth is not proportional to $S^{1/2}$ is no presented." Fig. 10 clearly indicates that the exponents of "S" varies fron 3/10 to 1/2 for the ripple bed. Mr. Lean might have overlooked the limitatio that Fig. 10 only gives ripple bed (or dune bed) up to d ≤ 20 mm beyond whicl dunes do not seem to form.

The writer agrees with Mr. Lean's suggestion to correlate the ripple siz with equivalent uniform roughness. To do this, however, one will encounte the difficulty that ripples and dunes have not yet been systematically define quantitatively.

Mr. Brush also questioned the applicability of the author's discharge formula to prototype alluvial channel and suggested a method for wall correctior The first question can be answered by Garde's work. The second question i that theoretically if R_b is truly known, there is only one possible solution fo the mean velocity. The scatter of the computed velocity data as compared t the measured data is more attributed to the incomplete criterion of boundar regimes than to the standard wall correction. Mr. Brush's suggestion on th technique for wall correction as well as using R instead is certainly accepte by the writer.

The presentation by Brush of σ and η in Eq. 1 merely indicates that σ ar η are also the variables for the study of the problem of alluvial channel rough

[4] "Laws of Turbulent Flow in Open Channels," by G. H. Keulegan, U. S. Nationa Bureau of Standards, Journal of Research, RP 1151, 21:707-741, December 1938.

ness. The effects of these variables on the resistance to flow still remain unsolved at the present time. Eq. 1 thus only presents the variables which govern the natural phenomenon of this particular problem. If the original function was written as Eq. 2(a) directly, it would erroneously imply that only those eight variables are important.

Mr. Brush also suggested that Eq. 7 may also be a good aspect of the problem on which to work. The disadvantage of Eq. 7 is that it contains two dependent variables; namely, V and h and it has been a difficult problem to determine h, particularly under field conditions.

However, the USGS[5] has recently developed a sonic sounder that makes it possible to precisely measure the spacing, amplitude, and velocities of the ripples and dunes under both laboratory and field conditions, making the suggestion feasible.

Messrs. Culbertson, Jordan, and Colby again used natural river data to test the validity of the authors' discharge formula. Their findings showed that the computed discharge over the flat bed in an alluvial channel is much larger than the discharge obtained from streamflow measurements. The reason is partly due to the inadequate criterion of the bed regimes, and partly due to the fact that the flat bed in natural channels may not be as smooth as that in the laboratory flumes. The discussers may find some of their answers from Garde's work on the new criterion of bed configurations.

Mr. Blanchet pointed out the important effects of the density and shape of particles on the resistance to flow. Data used by the writers do not cover a wide range of density of particles. For river hydraulics the density of particles usually varies only slightly. As to the shape effects, little is known today (1961). Engineers are forced to simplify the solution by making certain assumptions which may be tolerable for practical problems.

Messrs. Dawdy and Carter pointed out that the Blasius formula is valid only in the range of Reynolds number less than 80,000. For a Reynolds number of 00,000, where most flume data lie, there is a difference of 6% in f by the following two equations:

$$f = 0.0054 + \frac{0.396}{R^{0.3}}$$

nd

$$f = \frac{0.316}{R^{0.25}}$$

Because of the extreme complexity of this problem, the error thus introduced by the approximation should be allowable.

The points used to define C_a in Fig. 11 are the average of a series of runs r a given size of particles.

Messrs. Culbertson and Nordin made a different classification for the bed nfiguration based on data from Rio Grande River possessing a true sand d. Their classification of regimes gives a better result on the estimated locity. It is the writers' opinion that the disadvantage of such classification that it contains mean velocity which is an unknown variable. Accordingly, rial and error technique is needed for the computations.

[5] Sonic Depth Sounder for Laboratory and Field Use, USGS Circular 450, by E. V. chardson, D. B. Simonds, and J. Posahony.

Moreover, the plot V versus R does not reveal the effects of grain size and channel slope explicitly. A velocity equation according to Fig. A by Culbertson and Nordin can readily be written as follows:

$$V = C R^{\alpha}$$

To generalize the coefficient C is undoubtedly an extremely difficult task.

It is rather interesting to know that the writers' formula has been successfully applied to the Nile near Aswan Dam, according to Mostafa. Others show that a good agreement exists for the case of dune beds. In conclusion, the key role on the study of alluvial channel roughness problem requires an accurate classification of the bed regimes. Garde's new criterion should be tested further, and a systematic analysis of the available field data should be pursued.

Finally, the writer expresses his deep gratitude to those who have contributed the comments and criticisms. This closing discussion is especially written in memory of the first author, H. K. Liu, who unfortunately was fatally involved in an automobile accident in September, 1960.

HYDROLOGIC STUDIES BY ELECTRONIC COMPUTERS IN TVA[a]

Closure by W. M. Snyder

W. M. SNYDER,[7] M. ASCE.—The discussion by Kovner contains valuable ontributions defining additional potential computer applications in hydrology. he reference to direct digital recording and to the general trend toward data itomation are indicative of the evolutionary and challenging status of hydrol- gy.

Availability of computers has opened the door to many new methods of anal- sis. Full utilization of computer capability should include an evaluation of ıalytical methods currently in common practice. It may well be that a re- ersion to basic concepts is indicated and that practices common to an earlier atus of hydrology will become outmoded. Whether this is an admission that e old techniques do not always work or whether it is a response to a new sit- ıtion is a debatable point.

Kovner's comment that the method of time-definition of the rainfall pattern ıd explicit expression of a peak-rainfall relation represent progress in hy- ologic analysis is gratifying. It is agreed that the breakdown into periods is bitrary. Theoretically, there should be improvement in prediction of storm aks or volumes with more precise definition of rainfall pattern. However, point of diminishing returns is soon reached. The rate of build-up of the mber of terms in the prediction equation can be fantastic, and the regression lution for large numbers of coefficients of non-independent terms becomes satisfactory.

The final point of Kovner's analysis concerns an ultimate goal of a contin- us flow of data to a computer. While so-called on-the-line computer appli- tions may be in the distant future in hydrologic studies, a more immediate d necessary step which could lead toward continuity in flow of data appears be the development of continuous functions of hydrologic processes.

The use of arbitrary time periods in the peak-prediction equation was sim- a device by which a continuous process of rainfall to runoff conversion ring a storm was broken into a "stepped function." Some such device is cessary to produce linear coefficients which can be evaluated by the method least squares. However, there is available a method of non-linear least uares. Parameters of functions which are in non-linear form, and which nnot be linearized by logarithmic conversion, can be computed by the method successive approximation. Continuous functions containing relatively few rameters can then be substituted for the large number of ordinates of a epped function."

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[a] February 1960, by W. M. Snyder (Proc. Paper 2362).

[7] Formerly Head, Hydrology Sect., Hydr. Data Branch, TVA, presently Staff Research drol., Tributary Watersheds Program, TVA.

An example of the preceding analysis which come immediately to mind concerns the unit hydrograph principle. Unit hydrograph ordinates are a form of discrete representation of a continuous process of time-distribution of runoff into streamflow pattern. Perhaps some function such as a gamma function containing two or three non-linear parameters can be used in place of unit hydrograph ordinates. Such a function should be amenable to mathematical manipulation, allowing development of a comprehensive hydrograph model which also contains parametric expression for characteristics of individual watersheds. Applications of non-linear least squares, coupled with component analysis under multivariate statistical procedures, are now under study, and certain preliminary applications are being programmed for electronic computation.

A full explanation of, and reasons for the 4 x 4 Latin square design of the Western North Carolina Research Project are inappropriate in the closure of the discussion of electronic computers. A partial rebuttal to some of Kovner's statements are necessary, however, in fairness to the reader. First, one may question whether the concept of treatment versus control is standard practice in watershed research. What is standard for one agency may not be standard for another. TVA has conducted several watershed research projects. Few of them were "paired" watershed experiments. In fairness to Kovner, it should be stated that the treatment and control concept was the first design tried for the Western North Carolina Research Project. Two watershed pairs were set up, but after a period of calibrational observations, it was found that the watersheds were not paired in the sense that one could serve as a predictor for the other. Some new design was necessary.

This research project is conducted in cooperation with North Carolina State College, in Raleigh. H. L. Lucas of the Department of Experimental Statistics of the College proposed the Latin square design. It was a replacement for the treatment and control concept, and it is statistically more efficient than that concept. The beautiful symmetry, efficiency, and simplicity of a 4 x 4 arrangement of experimental data can be found in any statistical text covering experimental design.

Kovner may be implying that there was belated realization of the necessity for adjustment for varying meteorological conditions. This is not so. Lucas in his initial presentation of the plan demonstrated the manner in which this adjustment could become a powerful accessory to the basic design. Study by treatment and control is limited to a simple comparison of peaks or volumes with the influence of important meteorological variables determined only by some process of classification. Any other method based on inclusion of these variables in the regression of treatment on control would be an admission that meteorological adjustment is also necessary under that concept. However, by incorporating an adjustment procedure in the comprehensive plan based on the 4 x 4 Latin square, there is the possibility for comparison not only of peaks and volumes but of how the relationship of rainfall and peaks, for example, is influenced by watersheds, by covers, and by carry-over effect. This is accomplished by analysis of variance or regression coefficients in 4 x 4 arrangement. Various other statistics can also be computed for the sixteen data sets and studied by analysis of variance.

The peak and volume equations that were presented to illustrate regression studies by electronic computer are thus neither a procedure of expedience nor are they in final form. The effect of cultural treatment of the watersheds de

pends on composite analysis of sixteen data sets. The equations are adjustment devices which, after all data are collected, must be fitted in identical form to each of the sixteen sets. Until such time, the Western North Carolina Research Project with its 4 x 4 Latin square design will probably continue under study, pro and con.

SCOUR AT BRIDGE CROSSINGS[a]

Closure by Emmett M. Laursen

EMMETT M. LAURSEN,[39] M. ASCE.—The geographical distribution of the iscussers and the quality of the discussions indicate the widespread interest n and attest to the import of the problem of scour at bridge piers and abut- nents and similar obstructions in a stream. Without doubt the crucial issue aised by the analyses is the question of the effect of the velocity of flow on he depth of scour. The position of the writer is that there is a fundamental ifference in this regard depending on whether the approach flow supplies or oes not supply sediment to the scour hole; that under conditions of no supply, uch as a relief bridge, the velocity and the sediment size are important in de- ermining the depth of scour; that under conditions of supply by the approach low well above the critical tractive force, the velocity and sediment size have ttle effect except insofar as they determine the mode of sediment movement.

The position of Joklekar, Chitale, Thomas, Ahmad, Blench, and Bradley either explicitly or by reference) is that the depth of scour is proportional to e two-thirds power of the discharge per unit width. Because the discharge er unit width is the product of the velocity and the depth, for a given geome- y (including the depth of flow), the depth of scour should vary with velocity. lternately, one may rewrite the Poona equation so that $d_s/b = f\left(y_0/b,\ F\right)$. gain indicating that, for a constant geometry, the depth of scour is a function f the velocity. Although the position of Tison and Romita is not entirely clear, ne may infer from their analyses that the velocity has an effect on the depth f scour. Interestingly, none of them seemed to stress an effect of sediment ize.

The case of the long contraction can be used to illustrate the fundamental ifference between clear-water scour and scour in a sediment-transporting ream. The merit of this case for illustrative purposes is that the flow con- tions and sediment-transporting competence and capacity can be evaluated ith relative confidence and agreement, especially if the complicating features the zone of non-uniformity and of the ripple and dune formation are disre- rded. If one now considers a contraction of some given geometry, two widths d a depth of flow, it is possible for the velocity of flow to be so small that ere is no movement of the sediment anywhere. At or below this velocity, that dependent on the sediment size, there will be no scour, and the flow will have as if there were a rigid bed. If the velocity is increased, but not to ch an extent that there is sediment movement in the uncontracted approach, e contraction will scour. The limit of the depth of flow (or scour) will be a locity (or tractive force) that will not move the bed sediment. For this case

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[a] February 1960, by Emmett M. Laursen (Proc. Paper 2369).

[39] Assoc. Prof., Dept. of Civ. Engrg., Michigan State Univ., East Lansing, Mich.

of clear-water flow in the long contraction, it is readily apparent that the velocity of flow and the sediment size, as well as the geometry, will affect the depth of scour. The similar argument for the case of sediment-transporting flow in a long contraction culminated in Eqs. 6, 7, and 8 in which the velocity and depth of flow enter only insofar as they affect the mode of movement.

The confused flow in the area of scour around a bridge pier or abutment cannot be as well described, but one can expect that qualitatively the effect of velocity and sediment size would be similar to that in the case of the long contraction. Thomas explicitly states that the Poona experiments that resulted in Eq. 22 were run "without sediment load." Eq. 22 rewritten so that

$$\frac{d_s}{b} = 4.05\ F^{0.52}\left(\frac{y_0}{b}\right)^{0.78} - \left(\frac{y_0}{b}\right) \qquad (29)$$

will approximate the curve through the Poona data in Fig. 17 with a Froude number of 0.2. The curve for a Froude number of 0.4 would cross the writer's proposed design curve at approximately y_0/b equal to unity and would rise to a value of d_s/b equal to 4.2 at y_0/b equal to 6. Higher values of the Froude number would indicate even greater scour. Chitale, after describing these same Poona experiments, cites further experiments which evidently involved uncontrolled sediment movement at the higher velocities: "In a few tests in which the upstream depth was less than the stable depth the upstream bed scoured and blanketed the scour pit around the pier. In such cases the depth of scour at the nose of the pier was measured just before deposition in the scour hole of sand from upstream occurred."

The data for the higher velocities from Chitale's Table 2 if plotted in Fig. 9 would fall between the design curve (drawn conservatively) and the analytic curve, Eq. 11, as a group of points with d_s/b between 1.19 and 1.33 and y_0/b between 0.91 and 1.23. Blench presents a relationship, Eq. 12, reduced from the Poona equation and based on the same experiments without sediment load, that would also pass through the Poona data in Fig. 17. Romita cites an increase in the depth of scour with an increase in velocity, but again the tests at Milan were run ". . . in such conditions that there was no general bed movement but only localized scour around the pier."

Correspondence with Ahmad established the fact that his experiments were conducted so that there was good general movement of the bed. However, he used the average depth of the stream to obtain the depth of scour by the writer's proposed method and admits that a small angle of approach between the flow and pier can have a considerable effect. A modest increase in depth of flow and a small angle of attack could easily account for the discrepancy between the observed and predicted depths of scour in Table 4.

Blench objects to the use of a total load equation for the development of Eqs. 7 and 8 on the grounds that it is bed load that is mainly effective in determining depth. He feels that the formula is rough, and implies that y is not the regime depth. Eqs. 7 and 8 are for the case of the long contraction. If there is to be mass conservation in regard to the sediment load, a total load relation must be used. The difference in depth of scour is not large for different modes of movement as can be inferred from Eq. 8 and seen in Fig. 2. The writer's sediment load relationships were used because they permitted the evaluation of this effect. Any other sediment - transport equation will give approximately the same results because the load is not treated absolutely but in ratio. The depth y is the regime depth in the sense that it is the equilibrium

depth that will not change with time, although it may not be equal to Lacey's regime depth that does not consider the effect of load. If the equilibrium depth does not obtain, one should also consider the first category of scour mentioned as "those characteristics of the stream itself."

The Milan experiments are cited by Romita as differing from the Iowa results with the depth of scour for zero angle of attack lying below the curve representing Eq. 11 on Fig. 9 and the effect of angle of attack being greater than indicated by Fig. 10. Because the conditions of the experiments were different, the Milan experiments being run without sediment load, one can only speculate on the reason for the discrepancy. The lesser scour noted at zero angle of attack may be because of the velocity effect of clear-water scour. Romita, in fact, mentions an increase in depth of scour with an increase in velocity. The greater effect of angle of attack may be related to the affect of turbulence level on the critical tractive force. The critical tractive force is not really the mean value of the shear that will just move the sediment particles (except in laminar flow), but the mean value of the shear when the maximum shear due to turbulent fluctuations will just move the sediment particles. In the disturbed, confused flow in a scour hole one might expect that the turbulence level would be greater than normal, and that the increase would be related to the degree of obstruction. Because the limiting condition for clear-water scour is the critical tractive force on the boundary of the scour hole, it is possible that the effect of geometry such as angle of attack would be magnified over the effect found for the case in which there is a sediment supply.

Similarly the considerable effect of shape found by Tison may be due to the lack of supply. A lesser velocity than used would probably have resulted in all scour depths being reduced, and if a small enough velocity were used, there might not have been any scour at all around the better shaped piers. Thus, the relative effect of shape would have been magnified even more. Tison's demonstration of the causation of secondary flow as a consequence of the horizontal curvature of the stream-lines and the vertical velocity distribution is sound albeit qualitative. How this argument can be extended to the drastic change in the flow pattern after a scour hole has developed, however, is difficult to envision.

Bauer's crossing design should achieve the two ends he obviously intends; limiting the depth of scour to insure the safety of the bridge, and permitting a controlled depth of scour to reduce the backwater. A few words of caution are in order with such a design. The rock layer must be composed of sufficiently large material so that it will not be scoured out during a flood; this is a clear-water scour problem that is not solved as yet (1961). The deeper the layer is placed the smaller the rock can be. The scour will be concentrated around the piers and abutments and may not extend over the entire crossing. An approximate idea of the lateral extent of the scour can be obtained by equating the cross-sectional areas of the arrested and unarrested scour holes.

Bradley's cautionary comments should be heeded by all engineers seeking to use the proposed methods of predicting scour. The relationships proposed have been tested at only one real bridge pier, and this at a site of simple geometry. His plea for field measurements cannot be seconded with too strong a voice.

TRAP EFFICIENCY OF RESERVOIRS, DEBRIS BASINS, AND DEBRIS DAMS[a]

Closure by Charles M. Moore

CHARLES M. MOORE,[1] M. ASCE.—In preparing the paper, the writers attempted to collect and summarize existing published information pertaining to trap efficiency of reservoirs and debris basins. In so doing they did not attempt to change or bring up to date any of the material in the published references even though, as Heinemann emphasized, some of the terms such as "sediment production" have been replaced in later years. It was emphasized in the original paper that there may be other published data of which the writers were not aware.

The writer agrees also with Heinemann's suggestion that "a date or a period of time should always accompany the trap-efficiency value given for any reservoir." His suggestion that trap-efficiency values would be more meaningful if expressed as follows, is well received:

(1) "The 50-yr trap efficiency of X reservoir is estimated at Y%."
(2) "The trap efficiency of X reservoir is Y% (1959)."

Kaser's intimate knowledge of the Colorado River facilities and his discussion added materially to the original paper. The information presented by Kaser no doubt will help those who may be considering use of the data. Kaser's analysis should aid materially in clearing up this situation.

It is regretted that a typographical error occurred in the definition of trap efficiency in the original paper. As emphasized by Heinemann, trap efficiency is expressed as the ratio between sediment accumulation and sediment inflow, not sediment outflow, as given in the original paper.

The writers are indebted to Heinemann and Kaser for their additions and corrections to the manuscript.

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[a] February 1960, by Charles M. Moore, Walter J. Wood, and Graham W. Renfro (Proc. Paper 2374).

[1] Head, Design Sect., Engrg. and Watershed Planning Unit, Soil Conservation Serv., Ft. Worth, Tex.

A COMPARISON OF STREAM VELOCITY METERS[a]

Closure by F. Wayne Townsend and F. A. Blust

F. WAYNE TOWNSEND,[12] and F. A. BLUST.[13]—The writers are pleased learn that results obtained by others agree with those obtained by the Lake rvey.

During a program of metering in the St. Clair River in April 1960, the Lake rvey secured further data comparing Price and Ott current meters. The easuring section in the St. Clair River was located at Port Huron, Mich. easuring procedures were the same as those used for the 1958 measurements

TABLE 5.—COMPARISON OF AVERAGE MEASURED VELOCITIES AT ALL PANEL POINTS

Meter (1)	Average Velocity, in Feet per Second (2)	Percentage variation of average Ott velocity from average Price velocity (3)
Price	3.784	
Ott	3.773	-0.29%

the lower Niagara River, with simultaneous revolutions of the meters, cable spended at 0.4 depth from a catamaran, being recorded. Flow conditions re similar to the Niagara River except that velocities were as much as 25% her than were velocities in the Niagara River. Table 5 summarizes the nparative data obtained from the St. Clair River measurements. The data own are based on a hundred and twelve simultaneous meterings with the two es of meters.

April 1960, by F. Wayne Townsend and F. A. Blust (Proc. Paper 2438).
2 U. S. Lake Survey, Corps of Engrs., U. S. Army, Detroit, Mich.
3 U. S. Lake Survey, Corps of Engrs., U. S. Army, Detroit, Mich.

SEDIMENT PROBLEMS OF THE LOWER COLORADO RIVER[a]

Closure by W. M. Borland and Carl R. Miller

WHITNEY M. BORLAND,[22] M. ASCE, and CARL R. MILLER,[23] M. ASCE.—At the time the dams were planned for the Lower Colorado River, many of the engineers were concerned about the change in regime that would be instigated by trapping the heavy sediment load of the river in the reservoirs and what the effect of taking large amounts of water would have on the regime of the river. Not only did knowledge exist at that time to study such problems, as explained by the writers, but it was applied to the river problems of the Lower Colorado River before Hoover Dam was constructed.

The writers cannot agree with the statement of T. Blench, F. ASCE, that there is a deplorable lack of opportunity in north American colleges to study river behavior, stable channels, and transport of sediment. To cite three colleges whose curricula is relevant and with the writers are quite familiar: the California Institute of Technology, Pasadena, Calif.; the University of California at Berkeley; and Colorado State University, Fort Collins, Colo. There are no doubt many others. Not only are courses taught on fluvial morphology, river regimen, and river control, but engineers from these schools are active in determining what problems are facing the planning and design engineers and are helping to solve them. The following comments refer to itemized questions in Blench's study.

1. Blench's first point on which he desires elucidation concerns Fig. 5, that shows average size distribution of transported sediment at total load stations on the Lower Colorado River and would show no bed size change with time. Three of these stations are placed where, in general, the river channel is in equilibrium, neither scouring or filling, and would not be expected to show bed size change with respect to time. The fact that a degrading river will uncover coarser sediment and, in general, its bed material will coarsen with respect to time, is well illustrated by regularly published reports of Colorado River channel work and investigation.[24] Both Figs. 12 and 13 of the cited report show the changes in bed material that have occurred with time at several river cross sections. Of particular interest is that for Section 28 immediately below Davis Dam, that shows that the bed material for 50% size has increased

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[a] April 1960, by W. M. Borland and C. R. Miller (Proc. Paper 2452).

22 Head, Sedimentation Sect., Hydr. Branch, Div. of Proj. Investigations, Bur. of Reclamation, Denver, Colo.

23 Asst. Chf., Watershed Tech. Research Branch, Soil and Water Conservation Research Div., Agric. Research Service, U. S. Dept. of Agric., Beltsville, Md. (Formerly with the Bur. of Reclam., Denver, Colo.)

24 "Report of River Control Works and Investigation - Lower Colorado River Basin - Calendar Years 1952 through 1956," U. S. Bur. of Reclamation, U. S. Dept. of Interior, Boulder City, Nev., November, 1957.

from 0.2 mm in 1938 to 4.0 mm in 1956. The general coarsening of the bed material is similar for River Cross Sections 38 and 33 below Parker Dam.

The All-American Canal, being designed for channel stability, as a canal, is not subject to variation in flow or the variation in sediment that it must transport. Therefore, it is quite dissimilar, and its behavior is quite different from that of the Colorado River.

For the Colorado channelization, a detailed study of the likely degradation was made using the results of the deep bed sampling, sediment transport equations, incipient motion criteria, and channel hydraulic characteristics. The present source of the sediment load is the bed and banks, primarily the banks, at which extensive quantities of material in the transportable sizes are available. With control of the channel to a given width, the only source of sediment becomes the channel bed. The deep bed sampling reveals the existence of coarse material that will armor the bed as the fines wash out. Admittedly, the 6-in. armor layer assumption is open to question. However, it is not merely a wild guess. Studies by Lane, the writers' experience, and actual field observations indicate that this armor layer assumption is not too much out of line. Furthermore, the analysis made indicates that it matters little whether the armor thickness was a little less or more, or that the required size of armor material was smaller or greater.

2. The work of L. B. Leopold, F. ASCE, and T. Maddock, F. ASCE, proposed[25] generalized formulas for the hydraulic characteristics of channels and sediment sizes and amounts. It was then necessary to substitute data collected from several streams into these formulas to evaluate coefficients and exponents. Two reports are available that show such a relationship.[26,27]

The Colorado River at Yuma is controlled by a conglomerate and granite formation and, thus, has not formed a channel in its own alluvium; therefore, it does not conform to laws for rivers flowing in their own alluvium.

3. The word "degradation" is sometimes used to cover both vertical and horizontal scour. The problem with the Colorado River in the Needles, California area and Cibola Valley was lateral or horizontal scour and a resultant effect of aggradation at the lower end of the reach. Without channelization and bank control the sediment contribution from lateral degradation would have continued for many years, creating a perpetual aggradation problem above Lake Havasu and above Imperial Dam. Actually, the channelization has done little to change the slope of the river. A construction alinement that would give a slope to minimize degradation in the channelized section would involve excessive excavation, untenable curves, greater riprap yardage requirements, and interfere with existing and projected valley land uses. One of the benefits of channelization, as described, was to lower the valley groundwater table, and additional degradation is not detrimental to this purpose.

4. Little meandering is anticipated. In fact, the design is really one of meander control. It is expected that some additional riprap will be required

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25 "The Hydraulic Geometry of Stream Channels and Some Physiographic Implications," by Luna B. Leopold and Thomas Maddock, Jr., Geol. Survey, Professional Paper No. 252, 1953.

26 "Design of Stable Canals and Channels in Erodible Material," by P. W. Terrell and W. M. Borland, Proceedings, ASCE, Vol. 82, Proceedings-Separate 880, February 1956.

27 "A Plan of Channel Erosion Control, Fivemile Creek, Riverton Project, Wyoming," Sedimentation Sect., Proj. Planning Div., Bur. of Reclamation, Denver, Colo., April 1953.

and maintenance allowances have been made for this. Initially, 1-1/2 cu yd of riprap per linear foot of bank are placed along the edge of the channel alinement at anticipated vulnerable reaches (Fig. 11 for size analysis).

5. The writers believe Blench refers to Leopold and Maddock rather than Leopold and Wolman. As explained previously, the Colorado River at Yuma, Ariz. does not flow in a channel of its own alluvium, but is controlled by a formation of conglomerate and granite.

6. Item 5.

7. Dominant discharge was examined in several ways, one of which was the Blench equation for dominant discharge. It is not too difficult to bracket dominant discharge on the Lower Colorado River because of the extensive regulation by reservoirs. The figures selected were primarily based on the study of the daily flow releases during the years and the projected releases for the future.

8. Blench's regime theories were applied, and the width obtained was approximately 300 ft. It is believed that width requirement for stability, with negligible transport and friable banks, increases with bed size as indicated by most theories, rather than decreases as shown by Blench's equations. Also, with Blench's theory, there is considerable trouble in selecting the proper B and F factors to be used.

9. Suspended sediment transport is difficult to simulate in a model and it is doubtful if regime theory can be used to scale suspended sediment transport. Much of the transport of the Colorado River is of the suspended nature. A model of a river reach of this length would be quite large and costly, especially if an effort was made to keep the scale large enough to get some semblance of similitude for suspended sediment transport. An analytical approach to the design of the channel allowed us to use various theories and empirical approaches to derive a number of independent answers that can be compared and evaluated. The final selection can be based on experience, our evaluation of practical considerations, and judgment.

WATER EDDY FORCES ON OSCILLATING CYLINDERS[a]

Discussion by Phillip R. Smith

PHILLIP R. SMITH.[12]—The results of the study presented in this report rify that under certain conditions the effect of interference between neighring cylinders in unsteady flow is similar to the effect in steady flow; that the drag of a cylinder directly behind a neighbor is less than if it were not ielded, and the drag of a cylinder placed along side a neighbor is increased. e range of unsteady flows to which these results are applicable is restricted cause of difficulties inherent in predicting resistance in any unsteady flow.

A consequence of the use of a pendulum to model the horizontal components particle velocities in progressing ocean waves is the absence of a vertical locity gradient. Requiring that the velocity at the bottom of the prototype ve be approximately 90% of the surface velocity yields a wave length to depth tio[13] of 14. The results are thus restricted to deep water waves.

Because both the test cylinder and the interfering cylinders were attached the same pendulum, applicable prototype wave lengths must necessarily be eat compared to cylinder separation. If the velocities within a wave at trail- cylinders are not to differ significantly, the cylinder separation in the proype should be less than one-eighth of the wave length. In the study, cylinder paration ranged from zero to eighteen times the cylinder diameter. Hence, propriate minimum wave lengths for the results to be applicable would be proximately 10 times to 100 times the cylinder diameter. The upper limit wave length is of course, infinity.

J. S. McNown, F. ASCE, and G. H. Keulegan have shown[14] that in unsteady w the ratio of the period of flow to the period for vortex formation as defined the Strouhal number is a fundamental parameter. This ratio can be referred as the Strouhal time, T_O/T_V. For a Strouhal time of 10 or greater the mo- is quasi-steady. If it is less than 0.1, separation and vortex effects are mportant. The Strouhal times for the unsteady flows of the study were about or the 2-in. cylinders and 10 for the 1/2-in. cylinder. Hence, the results sented in this study apply to unsteady flows that approach quasi-steadiness; is, flows in which inertial resistance is relatively unimportant compared elocity dependent drag. For lower Strouhal time, for example, from 1 to nertial resistance is of the same order as velocity dependent drag. Whether

November 1960, by Alan D. K. Laird, Charles A. Johnson, and Robert W. Walker oc. Paper 2652).

2 Research Asst., Engrg. Mechanics Dept., School of Engrg. and Architecture, Univ. ansas, Lawrence, Kans.

3 "Hydrodynamics," by H. Lamb, Cambridge Univ. Press, New York, 1932.

4 "Vortex Formation and Resistance in Periodic Motion," by J. S. McNown and G. H. egan, Proceedings, ASCE, Vol. 85, No. EM 1, January, 1959.

the effects of interfering cylinders on the resistance of a cylinder in this latter flow regime can be predicted from the results of this analysis is not clear. An extension of the study to cover the entire range of Strouhal times would resolve this question.

The prediction of resistive forces in unsteady flow is difficult even for non-interfering cylinders. As stated by McNown,[15] the resistive force of a stationary body in waves consists of three principle parts: The force due to velocity dependent drag, the force due to the pressure gradient necessary to accelerate the ambient flow, and the inertial forces due to acceleration of the fluid caused by the presence of the body. The corresponding bulk equation is

$$F_x = C_d\ D\ l\ \rho\ \frac{V^2}{2} + A_o\ l\ \rho \left\{ \frac{d(kV)}{dt} + \frac{dV}{dt} \right\} \quad \ldots\ldots\ldots\ldots (1$$

in which ρ is the density of the fluid, D denotes the diameter of the cylinder, l refers to the length of the cylinder, A_o is the reference area of the body, and V describes the velocity. The difficulty that arises in utilizing this equation is that the coefficient of drag C_d, and the coefficient of virtual mass, k, both depend on period. However, McNown and Keulegan[14] have presented evidence of a unique relationship between C_d, k and Strouhal time for circular cylinders. With such a relationship known, Eq. 1 can be integrated numerically.

The preceding method could be extended to interfering cylinders if the variation of C_d with k and if the variation of either C_d or k with time for the test cylinder were known for each particular cylinder arrangement. The large amount of experimental data obtained in the study presented in this report should furnish many hints as to the nature of these relations. The value of the work presented has not been questioned. Rather, the writer has presented ways in which the experimental procedures and the characteristics of wave motion limit the applicability of the findings.

15 "Drag in Unsteady Flow," by J. S. McNown, Proceedings, Ninth Internatl. Congress of Applied Mechanics, Brussels, 1957.

FLUME STUDIES OF FLOW IN STEEP, ROUGH CHANNELS[a]

Discussions by F. V. A. Engel, N. Rajaratnam, and Herman J. Koloseus

F. V. A. ENGEL.[9]—The authors have presented a stimulating approach to the important problem of a phenomenon related to the dissipation of energy in natural streams. Their investigation is also of great interest as the knowledge on flow patterns in the range of low Froude numbers, in particular with reference to Froude numbers close to unity, is scarce.

In rough channel flow, the authors distinguish between three regimes, tranquil, tumbling, and rapid flow. How is correlation obtained between these three regimes and flow in smooth open channels; that is, (a) tranquil flow, (b) rapid flow with an undular jump in the range of Froude numbers between 1.0 and 1.7, and (c) the jump with a surface roller or the plain, fully turbulent jump?[10] Where does the regime of the undular type of jump fit in with the classification given by the authors? Is tumbling flow some kind of an equivalent to the undular jump? In Table 1, criteria for the flow regimes in the first line rapid flow is distinguished from tranquil or tumbling flow, whereas in the last line of the table under General, tumbling is grouped together with rapid flow. Therefore, the question arises whether tumbling flow covers a transitional stage or is it part of rapid flow? Knowledge of the undular type of jump is still unsatisfactory in so far as most experimental observations are related to small scale models with scale effects. Inter alia boundary layer effects may account for this particular type of jump. As an example in the investigations by P. Böss[11] the Reynolds number range was close to 5,000. Could the authors indicate in which Reynolds number range their experiments were conducted?

Furthermore, it would be useful to know the Froude number range over which tumbling flow was observed. From Fig. 11 one might conclude that the Froude number to tumbling flow is constant and is

$$F = 0.83$$

as the relation given by the authors can be rearranged resulting in the Froude number. If tumbling flow is a transitional stage before the beginning of rapid flow, it may be expected that it covers a range of Froude numbers and is not merely characterized by a single value.

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[a] November 1960, by Dean F. Peterson and P. K. Mohanty (Proc. Paper 2653).

[9] Cons., Workington, Cumberland, England.

[10] "Experiments of the Flow of Water From a Reservoir Through an Open Horizontal Channel II. The Formation of Hydraulic Jumps," by A. M. Binnie and J. C. Orkney, Proceedings, Royal Soc. of London, Series A, Vol. 230, 1955, p. 237.

[11] "Berechnung der Wasserspiegellage," by P. Boss, Forschungsarbeiten VDI No. 284, Berlin, 1927.

Eq. 24, which is used as the first criterion in Table 1, may be ambiguous. Non-dimensional groups of dynamics like the Froude number or the Reynolds number are often used in a modified form. However, the writer[12] has shown that even simple mathematical operations with the genuine groups of dynamics are restricted. It is necessary to present some kind of logical reasoning why the expansion of the Froude number with an area ratio Y_1/Y_2 is a permissible operation. It should be shown whether this operation is limited to certain conditions or if it is generally applicable.

Inserting a few numerical values in Eq. 24 may demonstrate the case. Assuming the Froude number $\mathsf{F} = 1$, a necessary condition for rapid flow, and inserting for the area ratio 0.9 and 0.3, respectively, would result in 0.81 and 0.09. Even though in both cases the Froude number is unity, insuring rapid flow, the value 0.09 is definitely much smaller than 0.5. Therefore, in accordance with Table 1, the latter case would be in the range of tranquil flow, that contradicts the criterion established by the authors. It may be necessary to limit the values K/Y_1. Could the authors amplify Fig. 4 to include the limit of tranquil flow showing the onset of tumbling flow? The same applies to Fig. 11. It would be useful if they could show the band width of tumbling flow plotting the curves separating tumbling flow from the regimes of rapid flow and tranquil flow.

From Eqs. 24 and 6 two limiting points may be obtained. The writer assumes that the limit of tumbling flow could be obtained from Eq. 6 in rearranging the latter equation as

$$q/Y_1 = V_1 = 0.83\sqrt{g\,Y_1} \quad \ldots\ldots\ldots\ldots\ldots \quad (24)$$

Inserting the corresponding value $\mathsf{F} = 0.83$ in Eq. 24 yields

$$\left(0.83\ Y_1/Y_2\right)^2 = 0.5$$

with $Y_2 = K + Y_1$

$$\frac{1}{\left(1 + \frac{K}{Y_1}\right)} = 0.85$$

A limiting value of tumbling flow that may be plotted in Fig. 4 would be

$$\mathsf{F} = 0.83 \text{ and } \frac{K}{Y_1} = 0.178$$

For rapid flow the corresponding coordinates are

$$\mathsf{F} = 1.0 \text{ and } \frac{K}{Y_1} = 0.43$$

The comparison with three dimensional conditions regarding relative roughness spacing in closed conduits, as investigated by Morris, with the two-dimensional case, as investigated by the authors, appears to be permissible in the regime of tranquil flow. Do the authors consider that it is also applicable to regimes of tumbling and rapid flow? In case of rapid flow with fairly large

12 "Non-dimensional Groups as Criteria of Process Plant Dynamics," by F. V. A. Engel, The Engineer, Vol. 206, 1958, London, p. 479.

spacing of the roughness elements hydraulic jumps would occur successively, a phenomenon that has no counterpart in closed conduits. Under those conditions the comparison would fail. However, with closely spaced roughness elements skimming in open channels flow may occur. Therefore, it may be necessary to limit the statement regarding the distance L of the roughness elements spacing. Also, as the authors apply their results to natural streams, it appears to be essential that some limits or a closer specification of the length of the crest of the roughness element in the direction of flow be given. From the illustrations and the dimensions given for the experiments the roughness elements could be classified as short structures. In weir flow,[13] it is known that the geometry of the weir entrance section and also the length of crest in direction of flow are pertinent features. It may be that the phenomenon of tumbling flow is directly connected with the relation between Y_1 and the length of the weir crest in direction of flow. It would be useful if the authors would comment on this issue.

In a final section, the authors suggest a tumbling-flow flume as means for stream-measurement. What accuracy do the authors expect? What would be the upper and lower limits of the measuring range, as there is always the chance that over a large range, the flow regime may change from rapid flow to tumbling flow and even for low heads to tranquil flow. The discharge coefficient may alter considerably under such conditions.

N. RAJARATNAM.[14]—The classification of the flow into the three regimes is interesting. The criteria formulated in Appendix I for the delineation of the three flow regimes is quite approximate because of the number of approximations involved, as for example:

1. The assumption of hydrostatic pressure distribution, especially at the section of measurement of y_1 is questionable because of the appreciable flow curvatures involved.
2. The drag coefficient has been assumed to be two, but in supercritical open channel flows, it is possible that it might vary with the Froude number of the flow.
3. It is well known that the limit of modularity varies with the shape of the weir contrary to the adoption of a constant value of 0.85 by the authors.

All these points can be clarified only by systematic experiments and in the absence of definite information on some of these aspects, the approach adopted by the authors is quite reasonable.

The writer is interested in the energy loss in tumbling flow. For uniform tumbling flow with bar roughness, the energy loss for each cycle is due to the friction loss in the length L and the loss in the jump. Unless the spacing L is considerably large, the friction loss is likely to be small compared to the loss in the jump. The hydraulic jump in the tumbling flow regime is different from the normal jump on a sloping floor because of the introduction of the form drag because of bar roughness. But realizing that even the case of the normal jump on a sloping floor has not been solved quite successfully and that definite information on the drag forces in flows with predominant gravity effects is not

13 "Weirs for Flow Measurement in Open Channels, Part 3 - Geometry of Weir Design, It's Influence on Discharge Characteristics," by F. V. A. Engel and W. Stainsby, Water and Water Engineering, Vol. 62, No. 784, 1958, p. 238.

14 Senior Scientific Officer, Civ. and Hydr. Engr., Indian Inst. of Science, Bangalore, India.

available, it is permissible at least as a first approximation to adopt Kindsvater's equation[15] for the (case-3) jump for the jump in tumbling flow.

Referring to Fig. 17, if d_1 is the normal depth before the jump (it is possible that, like y_1, d_1 also is likely to be independent of S_o and K/L for tumbling flow. Further, it can be written that $d_1 = \theta y_1$, d_2, the vertical depth just before the bar roughness, then is

$$\frac{d_2}{d_1} = \frac{1}{2 \cos \alpha} \left\{ \sqrt{1 + \frac{8 F_1^2 \cos^3 \alpha}{1 - 2 \phi \tan \alpha}} - 1 \right\} \quad \ldots\ldots\ldots (30)$$

in which α is the angle of the channel with the horizontal,

$$F_1 = \frac{\overline{V}_1}{\sqrt{g\, d_1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (31)$$

($\overline{V}_1$ being the velocity at the beginning of the jump) and ϕ is a factor depending on F_1 and α.

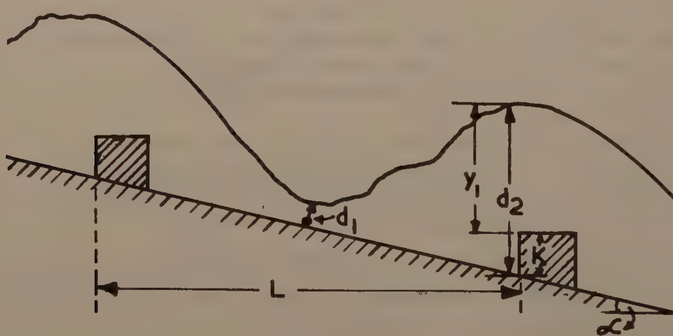

FIG. 17.—DEFINITION SKETCH

The loss of specific energy E_L is given by the expression

$$E_L = (d_1 \cos \alpha - d_2) + \left\{ \frac{\overline{V}_1^2}{2 g} - \frac{\overline{V}_2^2}{2 g} \right\}$$

$$= d_2 \left[\left(\frac{\cos \alpha}{\frac{d_2}{d_1}} - 1 \right) + \frac{1}{2} \frac{F_1^2}{\left(\frac{d_2}{d_1}\right)^3} \left\{ \left(\frac{d_2}{d_1}\right)^2 - 1 \right\} \right] \quad \ldots\ldots\ldots (32)$$

In Eq. 32

$$d_2 = K + y_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (33a)$$

15 "The Hydraulic Jump in Sloping Channels," by Carl E. Kindsvater, Transactions ASCE, Vol. 109, No. 2228, 1944, p. 1107.

$$d_1 = \theta\ y_1 \qquad (33b)$$

and

$$\mathbf{F}_1 = \frac{V}{\sqrt{g\ \theta^3\ y_1{}^3}} \qquad (33c)$$

For any given case of tumbling flow, y_1 and θ (from further analysis of the collected data) can be obtained and hence the energy loss.

If S_o and n_o are, respectively, the slope and roughness of the steep rough channel, without any bar roughness the normal depth of flow y_o is given by the equation

$$y_o = \left\{\frac{V\ n_o}{1.486\ S_o{}^{1/2}}\right\}^{3/5} \qquad (34)$$

[A controversy exists (as of 1961) on the extension of the Manning's formula into the supercritical range. Experimental information[16,17,18] (that are by no means conclusive) are available for supporting both sides of the argument.] By the introduction of the bar roughness, the tumbling flow is created and hence more energy loss. A "virtual channel" of a slope s_i and roughness n_i (without the bar roughness) can now be conceived that will still give uniform flow with y_o as the normal depth and also the same energy loss as the tumbling flow. Such a slope S_i will directly indicate the loss of energy per unit length of the channel. S_i and n_i can be shown to be given by the equations

$$S_i = \frac{E_L}{L} = \frac{(K+y_1)\left[\left\{\dfrac{\theta\ y_1 \cos\alpha}{(K+y_1)} - 1\right\} + \dfrac{V^2}{2\ g(K+y_1)^3}\left\{\dfrac{K+y_1{}^2}{\theta^2\ y_1{}^2} - 1\right\}\right]}{L} \qquad (35)$$

and

$$n_i = n_o\left(\frac{S_i}{S_o}\right)^{1/2} \qquad (36)$$

For the case of tranquil flow and rapid flow, to compute S_i and n_i, a precise knowledge of the energy dissipation occurring in the wake of the roughness elements (with predominant gravity effects) is necessary. Further the friction loss in the length L may also have to be considered.

For the description of the intensity of roughness

$$\lambda = \frac{(1-p\ t)^b}{L} \qquad (37)$$

would have been better because the vertical dimension K has already been considered as K/y_1.

Acknowledgments.—The writer is thankful to N. S. Govinda Rao for his encouragement in the preparation of this analysis.

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16 "Flow in a Channel of Definite Roughness," by R. W. Powell, Transactions, ASCE, Vol. 111, 1946, p. 531.

17 "Fluid Resistance in Water Flow of High Froude Number," by M. Homma, Proceedings, 2nd Japan Natl. Conf. on Applied Mechanics.

18 "Resistance Experiments in a Triangular Channel," by R. W. Powell and C. J. Posey, Proceedings, ASCE, Vol. 85, No. HY11, May, 1959.

HERMAN J. KOLOSEUS,[19] M. ASCE.—The effect of boundary irregularities has been studied by many investigators. In much of this work, the ratio of the height of the irregularity to the depth is small. However, as emphasized by Peterson and Mohanty, there are situations in nature in which the magnitude of this ratio is large instead of small. For this reason, their investigation is important and of interest.

The report of Peterson and Mohanty raises a question of definition. When should a boundary irregularity be considered as a roughness element and when as a change in channel cross-section? Some boundary irregularities are so small that the over-all planeness of the free surface is not affected by them. The writer is inclined to call these roughness elements. There are others for which a definite relationship exists between the irregularity and deviations of the surface from a plane. These are viewed by the writer as channel cross-sectional changes. With regard to the first type, experimental data indicate that the flow (when stable) is independent of the Froude number, while for the second the flow and the deviations of the free surface might reasonably be expected to be associated with a Froude number. It is possible to classify all boundary irregularities as roughness elements whether or not they had an influence on the planeness of the free surface. However, differentiation between a roughness element and a cross-sectional change would carry with it connotations concerning the planeness of the free surface and the Froude number. According to this classification, it is judged, mainly on the basis of the magnitude of K/Y_1 that much, if not all, of the work of the authors would be classified as flow through cross-sectional changes. Most of the results are expressed as a function of the Froude number through Y_c $\left(Y_c = Y_1 \mathbf{F}^{2/3}\right)$.

The suggestion that a tumbling flow reach might be used for evaluating the discharge is an interesting application of the characteristics of this type of flow. Here, the boundary irregularity reduces the cross section to such an extent as to form a control. The range of applicability of a wide (relative to depth) structure of this nature is limited because for a given slope and boundary irregularity rapid flow would replace tumbling flow beyond a certain discharge. This change in flow regime involves a change from a simple to a more complex stage-discharge relationship.

The authors have, as well they might, defined their Froude number in terms of the discharge and depth at the upstream edge of the boundary irregularity. At this point in the flow, the pressure distribution is not hydrostatic and in all likelihood a zone of separation exists. These conditions differ from those ordinarily associated with the Froude number. As a consequence, the usual connotations of the Froude number no longer pertain. These restrictions should be kept in mind when interpreting the results on the basis of the Froude number.

The remarks on roll waves were of special interest to the writer because he too has observed them in the laboratory. Unlike the authors, the writer, using cubical roughness elements, has found a fairly good correlation between the stability criterion[20] and the presence of roll waves. In the writer's experiments, the ratio of the height of the cubes to the depth was in most times

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[19] Hydr. Engr., U. S. Geological Survey, Iowa City, Iowa.

[20] "The Effect of Free-Surface Instability on Channel Resistance," by Herman J. Koloseus, thesis presented to State Univ. of Iowa at Iowa City, in August, 1958, as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

a small fraction of that of the authors. As a consequence, the writer believes that the flow was more uniform during the course of his experiments. He does not know if the discrepancy between these two sets of results can be attributed entirely to the difference in uniformity. An investigation of this point should prove of interest. It is hoped that the authors will continue their work, especially with cubes of mixed sizes.

FREQUENCY OF NATURAL EVENTS[a]

Discussion by Harold G. Lorsch

HAROLD G. LORSCH,[12] M. ASCE.—Of particular interest in the author's resentation should be the result that the probability of the n-year event being xceeded in an n-year period is approximately 0.64 rather than 0.5 as might e expected by the statistically uninitiated. This may be due to the nomenclaure in general use, such as "100-yr flood" or "50-yr storm." These signify verage recurrence intervals of 100 yr and 50 yr, respectively. They do not ignify 50% probabilities of the event being exceeded or not being exceeded in 100-yr or 50-yr period.

The writer believes that many designers are not aware of that distinction nd often incorrectly think of recurrence interval and 50% probability of ocurrence during the design period (taken equal to the recurrence interval) as eing identical. The curves in Fig. 6, which the author aptly calls "designrobability curves" and the numerical values of Table 3 should dispel that misonception.

The writer has plotted general curves similar to the author's Fig. 6 on log-g paper, using the data of Tables 2 and 3. These graphs (Fig. 10) allow a uick determination of the recurrence interval to be used for a given probabily of non-exceedance in the design period.

For example, in order to find the recurrence interval for which a design hould be dimensioned for 50% probability of exceedance (or non-exceedance) uring a design period of 20 yr, follow the vertical line through an abscissa of) to the curve for 50% probability of non-exceedance, for which point the reurrence interval ordinate reads slightly below 30. The exact value is 29.4 yr uthor's Table 3). If the probability of exceedance is to be 25% only, the probility of non-exceedance (which is used in Fig. 6) equals 1-0.25 = 0.75.

Repeating the preceding procedure but using the intersection with the curve r 75% probability of non-exceedance yields a recurrence interval of 70 yr 1 yr from the author's Table 3). The event having a 70-yr recurrence inrval should therefore be used as a design criterion, if a 25% probability of ceeding the design value in the 20-yr design period is desired.

It can be seen that the design-probability curves at constant probability of n-exceedance are straight lines except for short design periods. Any other

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a January 1961, by H. C. Riggs (Proc. Paper 2706).

12 Research Engr., Space Structures, Missile and Space Vehicle Dept. General Elecc Co., Philadelphia, Pa.

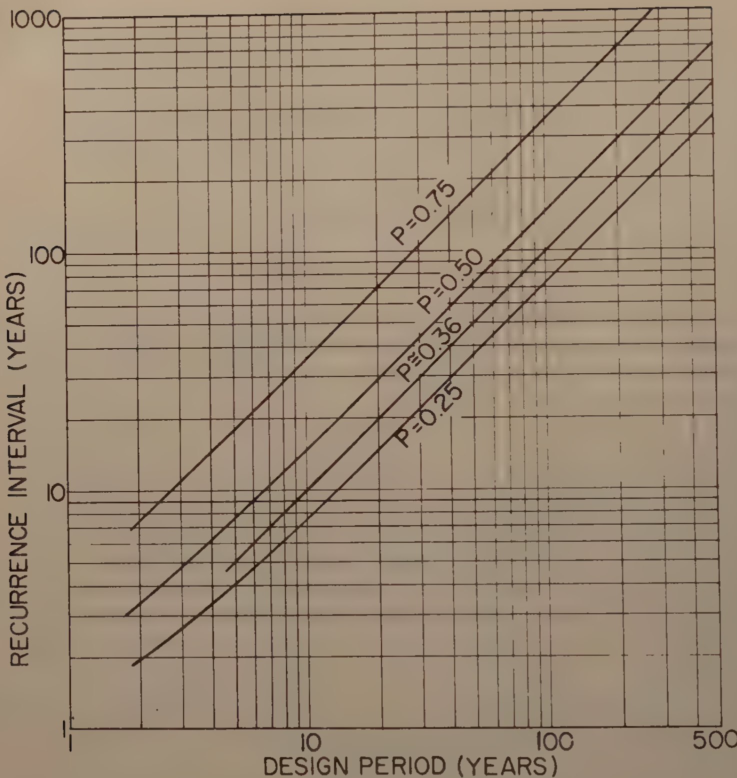

FIG. 10.—DESIGN-PROBABILITY CURVES

desired probability curves can easily be plotted by the use of the equation

$$p = \left(1 - \frac{1}{R.I.}\right)^{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad ($$

in which p is the probability of non-exceedance, R.I. denotes recurrence i terval (ordinate), and n refers to the design period (abscissa).

DEVELOPMENT OF A VARIABLE RELUCTANCE VELOCITY METER[a]

Discussion by M. B. McPherson

M. B. McPHERSON,[4] M. ASCE.—The miniature meter described appears to have some long-sought performance characteristics. The information given is quite adequate for a progress or development report. However, there are some questions that require answers before the versatility and potential value of the meter can be appraised fully. As a prologue to these questions, the development tests for two variable resistance meters will be reviewed.

B. B. Somervell[5] has described the performance of a free-spinning meter designed by G. B. Pegram for use in sea water. The final model had a four-blade propeller approximately 3 in. in diameter mounted within a 6-in. ring frame. Insulators were mounted on two opposing blade tips. A D. C. current was passed through two poles, set approximately 35° apart on the periphery of the blade tip circle, with the sea water serving as ground. Transit of an insulated tip past the poles caused a transitory increase in conductivity, observed as a periodic trace by means of a recording galvanometer. The triangular insulated blade tips, oriented normal to the propeller and poles, were so arranged that either an apex or a full side of a triangle was presented first to the poles. The resulting varied conductance defined direction of rotation for each half revolution through the sign of the output signal. Use of the two closely adjacent poles was found necessary for satisfactory resolution of the signal sign. "In so far as recording reversals of current is concerned, the meter has been successful. Its sensitivity has likewise been all that could be desired, readings as low as 0.08 ft per sec being secured." Towing tests were performed in a swimming pool, with the meter axis oblique to the direction of the carriage motion. "When inclined up to 20° in a positive direction, the Pegram meter gave little or no variation from the cosine curve; at 20° in a negative direction from 5% to 15% variation; - - -." A miniaturized version of the Pegram meter might be useful in laboratory work, but if the water conductivity would have to be raised, the corrosive capacity of the salt solution would be a deterring liability.

A later type of midget (or micro) meter,[6] designed for fresh water measurement, was also based on the resistance principle. A two blade propeller with 45° pitch and a brass wheel with twenty-four square-cornered teeth were mounted on a free-spinning shaft. Two diametrically opposed insulated brass

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[a] January 1961, by Iury L. Maytin, (Proc. Paper 2713).

[4] Prof. of Hydr. Engrg., Univ. of Illinois, Urbana, Ill.

[5] Discussion of "Effect of Turbulence on the Registration of Current Meters," by Brehon B. Somervell, Transactions, ASCE, Vol. 95, 1931, pp. 800-805.

[6] "Current-Meter for Rapidly Varying Flow," Dauphinois Hydr. Lab., La Houille Blanche Revue De L'Ingenieur Hydraulicien, September, October, 1950, pp. 574-577.

electrodes were mounted on the meter frame, within a few tenths of a mm from the outer periphery of the wheel circle. The wheel diameter was about half the blade diameter. In the illustrative photograph the teeth and slots appear to be square. The electrode tips appear to be about half the width of the teeth (or slots). While the propeller appears to be quite small the actual size is not mentioned. The "rotation of the current meter causes periodic variations in the resistance between electrodes; these variations are then recorded on a cathode ray oscillograph by means of the usual type of electrical circuit." It is "possible to measure - - - speed variations over a fraction of a revolution." One of the teeth "is slightly filed away so that the number of complete revolutions can be more readily counted when the test results are being studied. The cathode ray oscillograph signals are recorded on photographic paper." A "signal of known frequency is superimposed on the same band of paper by means of an electronic commutator. Calibration tests of the currentmeter show that it does not start to move until a certain threshold velocity is reached." (In Fig. 5 it appears that the VRVM meter has a similar inertial characteristic with no rotation up to about 0.05 fps.) The midget meter was subjected to an acceleration test, the blade was restrained with a steady flow past the meter, then suddenly released. In an example cited (but mean velocity not given) one-tenth of a second was required for the meter to reach full response from a stalled position, "thus proving that the meter is highly sensitive and of small inertia."

Both of these resistive type meters require a small clearance between insulator (or conductor) and probes (or electrodes). This limitation does not appear to be nearly as critical for the author's instrument.

The direction of rotation of the steel sprocket could be determined directly in the VRVM by noting the direction of the current induced in the coil. Only the Pegram meter indicated flow direction, but it was particularly adaptable to reversal because there was no gear wheel or sprocket on the propeller shaft to interfere with the flow. It would be interesting to see a calibration curve for the VRVM meter towed with the sprocket foremost. The meter might be advantageously adapted to oceanographic current measurements.

Diamagnetic substances, a classification which includes most nonmetals, have permeabilities barely less than one, the value for a perfect vacuum. However, air and water together with nearly all other fluids have magnetic permeabilities only approximately $1/10^5$ less than unity. This fact has been ingeniously used by the author in developing a device suitable for measurements i a wide variety of fluids. The gap between pole and sprocket is said to be occupied by air, yet the data presented is exclusively for water. It is eviden that the electrical system is incapable of distinguishing air from water, but th effect of fluid friction on the performance of the propeller in the two case might be quite different (Figs. 3 and 5). This question could be resolved b removing the towing tank in Fig. 4 and running at a few different velocities i still air. It is requested that the author give geometric details of the propelle and frame for future reference.

Alterations could be made in the specifications for the coil, permanent mag net and pole without affecting the frequency versus velocity calibration (Figs 5 or 6). However, the slope of the voltage output versus velocity curve of Fig 3 might be appreciably altered. It is presumed that 4 cps equal 1 rps.

Apparently the propeller used in the tests for Figs. 5 and 6 was 1 in. in di ameter, but another with a 0.25 in. diameter was also tried. To fabricate de

vices of this size requires a high degree of miniaturization skill. The smaller propeller "proved extremely responsive to the most minute fluid velocity changes." Perhaps the author could provide a more tangible indication of response to unsteady motion (such as, by means of an acceleration test).

Will the smaller meter register accurately velocities less than the 0.15 fps of Fig. 5? The method for reading the frequency directly for a steady towing velocity is quite obvious. For a turbulent flow it appears that only a visually interpolated average could be read from the frequency meter, and that instantaneous values would be available only if the frequency meter output was fed into a pen recorder for subsequent interpretation.

Because the output from the coil suffers a cyclic variation, are the millivolts in Fig. 3 mean values? Either the magnetic flux in Eq. 1 is in webers rather than maxwells, or 10^8 should appear in the denominator.

The author has developed a device that appears to be of significant potential use. Perhaps citation of the geometric details and composition of the magnetic circuit components, including an estimate of the field strength of the permanent magnet, would promote further development and more immediate use of this device. It would also be interesting to know the dimensions of the circular towing tank and the position of the meter axis relative to water surface and tank floor. Can the possibility of bias due to boundary effects on meter registration be safely ignored?

Ultimately, some sort of tests for oblique orientation should be made, similar to those described by Somervell.[5] Perhaps the reversible current feature of the VRVM could be better used with a modified design patterned after the Pegram meter, using a propeller with ferromagnetic tips and a hollow shaft to offset the added inertia. The probe would then need extend only to the blade periphery, the sprocket would be eliminated, and the shaft could be shortened. An adaptation of the VRVM might have some advantages over the Price meter with magnesyn compass, as used in estuary measurements.[7]

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[7] "Prototype Measurements of the Columbia River Estuary," by J. B. Lockett and A. Kidby, Proceedings, ASCE, Vol. 87, No. HY 1, January, 1961.

STREAM GAGING NETWORK IN THE UNITED STATES[a]

Discussion by J. C. Stevens

J. C. STEVENS.[13]—Correlation of temporary stations with a representative tworkof permanent ones will produce, for the temporary ones in most cases, sults that are sufficiently accurate for all practical purposes. Of course, the rrelation must be made with engineering nicety considering the major factors ferentiating the watersheds. However, there is no use "splitting hairs" in is process.

Where permanent stream channels are available, correlations can be made th considerable confidence, but ever changing channels in loose soil and sand ll give trouble as they always have unless provided with stable controls. thout them correlation is liable to be too erroneous for reliability.

One wonders if hydrology, hydrologic, and so on, are the proper terms to be plied to the subject matter of this paper. The writer was raised on hydro- aphy, hydrographic, and so forth, but believes both questionable terms ap- cable to the stream gaging program.

The latest book treating the subject in question is a monumental work.[14] reign references in this work are given in their original languages followed translations in English.

Most foreign languages use the equivalent of and derivations from hydrom- y when treating of this subject matter of this paper. In this country, how- er, they seem to have been studiously avoided. Webster's Unabridged Dic- nary, latest edition, defines hydrometry as "the art or operation of the hy- meter" which in turn, is defined as "a floating instrument for determining ecific gravity, especially of liquids," — a rather narrow application, but ibtless justified.

Hydrology is defined as "the science treating of water, its properties, phe- nena and distribution over the earth's surface. The term is used specifically the United States Geological Survey (USGS) with reference to underground er sources, as distinguished from hydrography, which is applied to surface er supplies and sources," a distinction which may well be questioned.

In the latest edition of Funk and Wagnall's Britannica Dictionary hydro- phy is defined as "the science of determining and making known the condi- n of navigable waters, charting rivers, coasts, etc." While in the new Funk Wagnall's Encyclopedia it is defined as, "the study of lakes, seas and other lies of water and particularly the study of stream flow . . . ." It is, how- r, silent on hydrometry and hydrology.

The writer is fully aware that the term "stream gaging" was adopted quite ly by the USGS Water Resources Branch in order that Congress and the

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a March 1961, by John E. McCall (Proc. Paper 2776).

13 Cons. Engr., Leupold and Stevens Instruments Inc., Portland, Oreg.

public could have no doubt what its appropriations were for. It is quite likely that Marshall O. Leighton, with USGS 1903-1913, was largely responsible for that term.

Hydrometry really means the science of metering water. It appears that hydrology, hydrologic, hydrological, etc., could well be replaced by hydrometry, hydrometric, hydrometrical, etc., as being nearer the subject matter intended. In Webster's is found "Hydrometric pendulum, an instrument . . . used to measure the velocity of a running liquid." What would a hydrographic or hydrologic pendulum be?

We might well profit by that example and be consistent enough to use hydrometry and its derivations where such are required, because there can be no such derivatives from "stream gaging."

FORECASTING RIVER RUNOFF BY COASTAL FLOW INDEX[a]

Discussion by H. C. Riggs

H. C. RIGGS,[12] M. ASCE.—Storage as snow of a large part of the annual precipitation in Columbia River basin provides a time delay between precipitation and runoff which makes reliable runoff forecasts possible. All forecasting procedures depend on indexes of the water in snow storage. Snow-survey data and winter precipitation have been used extensively and successfully in forecasting the summer runoff of high-elevation basins. The use of another index, the winter runoff from low-elevation coastal basins, is shown to provide forecasts of comparable reliability for Columbia River basin above The Dalles.

In addition to the major independent variable (the index of stored water) in each forecasting equation, indexes of modifying influences are often used. A great many such indexes can be devised and various combinations of these and one major independent variable can be used in different ways in the forecasting equation. As a result, many different forecasting equations have been developed. From this, it might be expected that the superiority of a particular set of indexes could be definitely established, but this is not so. It is relatively simple to make a fairly good forecast, but extremely difficult to improve it appreciably. For example, the authors' values for April-through-September runoff and for the coastal-flow index may be used to define, on log paper, a linear relation from which only five of the thirty-one points deviate by more than 10%. This approaches the quality of the authors' result using additional variables.

The Columbia River basin comprises many large, diverse, sub-basins. It seems remarkable that one index of available water in the basin, and indexes of additions and losses to that available water so closely represent physical conditions. The coastal-flow index used by the authors has the advantage of integrating conditions over a considerable area in contrast to precipitation or snow-survey indexes which are averages of spot observations. On the other hand, the coastal-flow index method requires the assumption that the relation between winter precipitation on the index coastal basins and on the Columbia River basin above The Dalles does not change appreciably from year to year. That this assumption is not completely acceptable may be seen by examining the maps of annual runoff in percentage of mean for Columbia River basin for the years 1928-45.[13]

The authors state that accuracy of measurement of streamflow is an inherent limitation of index methods of forecasting runoff. Streamflow volumes are

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[a] March 1961, by David M. Rockwood and Carlton E. Jencks (Proc. Paper 2780).
[12] Hydr. Engr., U. S. Geological Survey, Washington 25, D. C.
[13] "Annual Runoff in Columbia River Basin in Percent of the Mean," by C. C. McDonald and H. C. Riggs, U. S. Geological Circ. Vol. 36, 1928-1945, p. 2.

accurate within a few percent; they are far more accurate than any other element in the forecasting equation. Then how can the accuracy of streamflow measurement be a limitation?

The authors define their forecasting equation as of July 1 and apparently assume that the true regression weights of all the variables are thus obtained. Their equation is an empirical relation between arbitrarily-selected indexes and it has no more fundamental significance than an equation based on any other date. Alternatively, the authors might have used the known runoff at The Dalles for April and for May-June in place of their variables X_4 and X_5, respectively. Neither procedure should give better regression constants for use as of April 1 than would be defined by an empirical equation based only on information available on that date. Now look at the equation itself. For use on April 1, each of the variables X_4 and X_5 reduces to a product of the coastal-flow index and the appropriate constant (regression coefficient times a mean value). The affect of these variables on the April 1 forecast is limited to the variation in the coastal-flow index. Thus, by the authors' procedure, the coastal-flow index is used three times in computing the April 1 forecast.

Justification for the practice of developing a forecasting equation for dates after the beginning of the forecast period (such as a July 1 forecast of April through September runoff) is not apparent. On July 1, the needed forecast is of the runoff for July, August, and September, not for the period April through September. The former can, of course, be obtained by subtracting from the authors' forecast the known runoff April through June. However, it would seem possible to make a better forecast by using the desired runoff as the dependent variable and relating it to the known discharge on July 1 and to the rate of recession. The standard error (in acre feet) of the authors' July 1 forecast is much smaller than that for their April 1 forecast, but if these standard errors are computed as percentage of the mean July-through-September runoff and the mean April-through-September runoff, respectively, the July 1 standard error will be the larger.

Three bases are now available for forecasting summer runoff of Columbia River; winter precipitation, snow-survey data, and the coastal winter flow. None of these seems to stand out as superior. It is likely that significant improvement in forecasting results will await a better understanding of the time distribution of losses in the various parts of the Columbia River basin. As more reservoirs are created, this problem may increase in complexity.

VECTOR ASPECTS OF DYNAMIC SIMILARITY[a]

Discussion by W. D. Baines and J. F. Keffer

W. D. BAINES,[9] A. M. ASCE, and J. F. KEFFER.[10]—Conventional deriva-ions of the requirements of dynamic similarity are simplified to the point that calar quantities only, seem to be involved. The authors make a strong point hat the forces used in these derivations are vectors and must obey all of the equirements inherent with these quantities. Clearer statements of these facts n contemporary test books would remove much of the "magical" aura sur-ounding the subject.

It is surprising to the writers, however, that a more rigorous development as not been used. It may be properly assumed that the equations of motion escribe the flows described by the paper, in which viscosity and density are ne only fluid properties considered. In vector form, these equations are[11]

$$\rho \left[\frac{\partial \bar{u}}{\partial t} + \bar{u}\,(\bar{u}\ \nabla)\right] = - \rho\, h\, \nabla Z - \nabla p + \mu \nabla^2 \bar{u} \qquad (38)$$

nd

$$\nabla \bar{u} = 0 \qquad (39)$$

is the velocity vector and all other quantities are defined in the paper. Eq. 8, the well-known Navier-Stokes equation and Eq. 39, the equation of contin-ity are actually 4 scalar equations with 4 independent variables (3 velocity omponents and pressure) and 4 dependent variables (3 co-ordinates and time). lathematically it can be shown, that for laminar flows, with appropriate bound-ry conditions specified, that a unique solution exists. For turbulent flows, niqueness cannot be proven because the boundary conditions are not clearly efined. However, for the mean flow, the unique solution appears reasonable om physical considerations.

The terms in Eq. 38 have the dimensions of force/volume and can be made imensionless by dividing through by a term having these dimensions. It is onvenient for physical reasons to choose a divisor having a characteristic elocity magnidute U, a characteristic length L, and the fluid density ρ. The equired term is $\rho\, U^2/L$ and Eqs. 38 and 39 reduce to

$$\frac{\partial \bar{u}'}{\partial t'} + \bar{u}'\,(\bar{u}\ \nabla') = - \frac{\nabla' Z'}{F^2} - \nabla' p' + \frac{1}{R}\ \nabla'^2 \bar{u}' \qquad (40)$$

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[a] March 1961, by R. C. Kolf and W. L. Reitmeyer (Proc. Paper 2763).

[9] Assoc. Prof., Mech. Engrg., Univ. of Toronto, Toronto, Ont.

[10] Instructor in Mech. Engrg., Univ. of Toronto, Toronto, Ont.

[11] "Advanced Mechanics of Fluids," Hunter Rouse Ed., John Wiley and Sons, Inc., ew York, 1959.

and

$$\nabla' \, \overline{u}' = 0 \quad \text{.......................} \quad (41)$$

in which $\overline{u}' = \overline{u}/U$, $\nabla' = L\,\nabla$, $t' = U\,t/L$, $p' = p/(\rho\, U^2)$, $F = U/\sqrt{L\,g}$ and $R = U\,L\rho/\mu$.

If now the solution to Eqs. 40 and 41 is written in functional form, two dependent dimensionless parameters only exist, that is, F and R. Fixed values of F and R define a unique solution. For example, if the pressure drop between two fixed points in a flow field is desired, the solution is

$$\frac{\Delta p}{\rho \frac{U^2}{2}} = E = f(F, R) \quad \text{.................} \quad (42)$$

in which E is the Euler number defined by the authors. If the shear at a boundary point is desired, the shear co-efficient is

$$\frac{\tau_o}{\rho \frac{U^2}{2}} = C_\tau = f(F, R) \quad \text{................} \quad (43)$$

Similarly, the velocity at any point is uniquely determined by F and R. In particular, the vortex number for rotational symmetrical flow is no more than a statement of the general solution for velocity at a point. If in cylindrical coordinates, v is the θ-direction velocity the general solution is

$$\frac{v}{U} = f(F, R) \quad \text{......................} \quad (44)$$

Thus the vortex number is not an addition to the classical similarity criteria but an inherent requirement of the geometrical similarity.

It is unnecessary, therefore, to consider various force ratios as the authors have done. Logical deductions from the differential equations of flow result directly in the functional relationships. The writers prefer this more rigorous approach since for general situations one cannot be sure that the correct number of forces have been defined. By purely physical reasoning, it is possible to choose too many or too few forces.

The writers hope that their discussion will serve to assist the authors in the clarification of a subject which, although simple, has confused engineers since its introduction.

PROCEEDINGS PAPERS

The technical papers published in the past year are identified by number below. Technical-division sponsorship is indicated by an abbreviation at the end of each Paper Number, the symbols referring to: Air Transport (AT), City Planning (CP), Construction (CO), Engineering Mechanics (EM), Highway (HW), Hydraulics (HY), Irrigation and Drainage (IR), Pipeline (PL), Power (PO), Sanitary Engineering (SA), Soil Mechanics and Foundations (SM), Structural (ST), Surveying and Mapping (SU), and Waterways and Harbors (WW), divisions. Papers sponsored by the Department of Conditions of Practice are identified by the symbols (PP). For titles and order coupons, refer to the appropriate issue of "Civil Engineering." Beginning with Volume 82 (January 1956) papers were published in Journals of the various Technical Divisions. To locate papers in the Journals, the symbols after the paper number are followed by a numeral designating the issue of a particular Journal in which the paper appeared. For example, Paper 2703 is identified as 2703(ST1) which indicates that the paper is contained in the first issue of the Journal of the Structural Division during 1961.

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AUGUST: 2564(SM4), 2565(EM4), 2566(ST8), 2567(EM4), 2568(PO4), 2569(PO4), 2570(HY8), 2571(EM4), 2572(EM4), 2573(EM4), 2574(SM4), 2575(EM4), 2576(EM4), 2577(HY8), 2578(EM4), 2579(PO4), 2580(EM4), 2581(ST8), 2582(ST8), 2583(EM4)^c, 2584(PO4)^c, 2585(ST8)^c, 2586(SM4)^c, 2587(HY8)^c.

SEPTEMBER: 2588(IR3), 2589(IR3), 2590(WW3), 2591(IR3), 2592(HW3), 2593(IR3), 2594(IR3), 2595(IR3), 2596(HW3), 2597(WW3), 2598(IR3), 2599(WW3), 2600(WW3), 2601(WW3), 2602(WW3), 2603(WW3), 2604(HW3), 2605(SA5), 2606(WW3), 2607(SA5), 2608(ST9), 2609(SA5)^c, 2610(IR3), 2611(WW3)^c, 2612(ST9)^c, 2613(IR3)^c, 2614(HW3)^c.

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NOVEMBER: 2637(ST11), 2638(ST11), 2639(CO3), 2640(ST11), 2641(SA6), 2642(WW4), 2643(ST11), 2644(HY9), 2645(ST11), 2646(HY9), 2647(WW4), 2648(WW4), 2649(WW4), 2650(ST11), 2651(CO3), 2652(HY9), 2653(HY9), 2654(ST11), 2655(HY9), 2656(HY9), 2657(SA6), 2658(WW4), 2659(WW4)^c, 2660(SA6), 2661(CO3), 2662(CO3), 2663(SA6), 2664(CO3)^c, 2665(HY9)^c, 2666(SA6)^c, 2667(ST11)^c.

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FEBRUARY: 2726(WW1), 2727(EM1), 2728(EM1), 2729(WW1), 2730(WW1), 2731(EM1), 2732(SM1), 2733(WW1), 2734(SM1), 2735(EM1), 2736(EM1), 2737(PL1), 2738(PL1), 2739(PL1), 2740(PL1), 2741(EM1), 2742(ST2), 2743(EM1), 2744(WW1), 2745(WW1), 2746(SM1), 2747(WW1), 2748(EM1), 2749(WW1), 2750(WW1)^c, 2751(EM1)^c, 2752(SM1)^c, 2753(PL1)^c, 2754(ST2)^c, 2755(PL1).

MARCH: 2756(HY2), 2757(IR1), 2758(AT1), 2759(CO1), 2760(HY2), 2761(IR1), 2762(IR1), 2763(HY2), 2764(ST3), 2765(HY2), 2766(HW1), 2767(SA2), 2768(CO1), 2769(IR1), 2770(HY2), 2771(SA2), 2772(HY2), 2773(CO1), 2774(AT1), 2775(IR1), 2776(HY2), 2777(HY2), 2778(SA2), 2779(ST3), 2780(HY2), 2781(HY2)^c, 2782(HW1)^c, 2783(SA2)^c, 2784(CO1), 2785(CO1)^c, 2786(IR1)^c, 2787(ST3)^c, 2788(AT1)^c, 2789(HW1).

APRIL: 2790(EM2), 2791(SM2), 2792(SM2), 2793(SM2), 2794(SM2), 2795(SM2), 2796(SM2), 2797(SM2), 2798(SM2), 2799(EM2), 2800(EM2), 2801(EM2), 2802(ST4), 2803(EM2)^c, 2804(SM2)^c, 2805(ST4)^c.

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JUNE: 2828(SM3), 2829(EM3), 2830(EM3), 2831(IR2), 2832(SM3), 2833(HW2), 2834(IR2), 2835(EM3), 2836(IR2), 2837(IR2), 2838(SM3), 2839(SM3)^c, 2840(IR2)^c, 2841(HW2)^c, 2842(EM3)^c, 2843(ST5), 2844(ST5), 2845(ST5), 2846(ST5)^c.

JULY: 2847(PO2), 2848(SU2), 2849(HY4), 2850(PO2), 2851(HY4), 2852(PO2), 2853(SU2), 2854(HY4), 2855(PO2), 2856(PO2), 2857(PO2), 2858(SA4), 2859(SU2), 2860(SA4), 2861(PO2), 2862(SA4), 2863(HY4), 2864(HY4), 2865(HY4), 2866(HY4), 2867(HY4), 2868(PO2)^c, 2869(SA4)^c, 2870(SU2)^c, 2871(HY4), 2872(HY4)^c, 2873(SU2), 2874(SA4).

^{c} Discussion of several papers, grouped by divisions.

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